6.1
Continuity at a Point

Learning objectives:

- To study the concept of continuity of a function at a point and to present continuity test
- To study the types of discontinuities through examples And
- To practice related problems

Continuity at a Point

A continuous function is a function whose outputs vary continuously with the inputs and do not jump from one value to another without taking on the values in between. Several physical processes proceed continuously, and they are represented by functions of a real variable and have domains that are intervals or unions of separate intervals.

We study the continuity of a function at a point. There are three kinds of points to consider: ***interior points, left endpoint(s),*** and ***right endpoint(s).***

Definition: continuity at a point

A function f is ***continuous at an interior point*** $x = c$ of its domain if

$$\lim_{x \to c} f(x) = f(c)$$

Continuity at end points is defined by taking one-sided limits. A function f is **continuous at a left endpoint** $x = a$ of its domain if

$$\lim_{x \to a^+} f(x) = f(a)$$

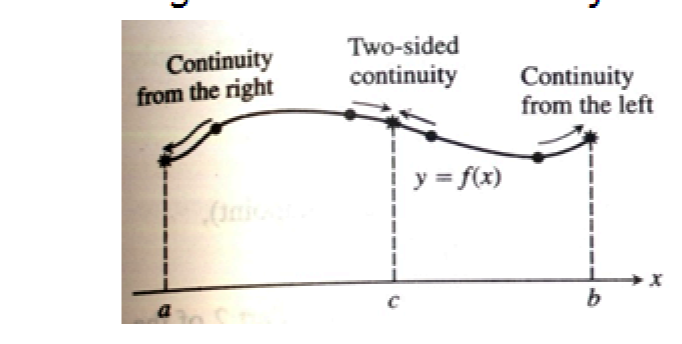
and **continuous at a right endpoint** $x = b$ of its domain if

$$\lim_{x \to b^-} f(x) = f(b)$$

In general, a function f is **right-continuous** at a point $x = c$ in its domain if $\lim_{x \to c^+} f(x) = f(c)$. It is **left-continuous** at c if $\lim_{x \to c^-} f(x) = f(c)$.

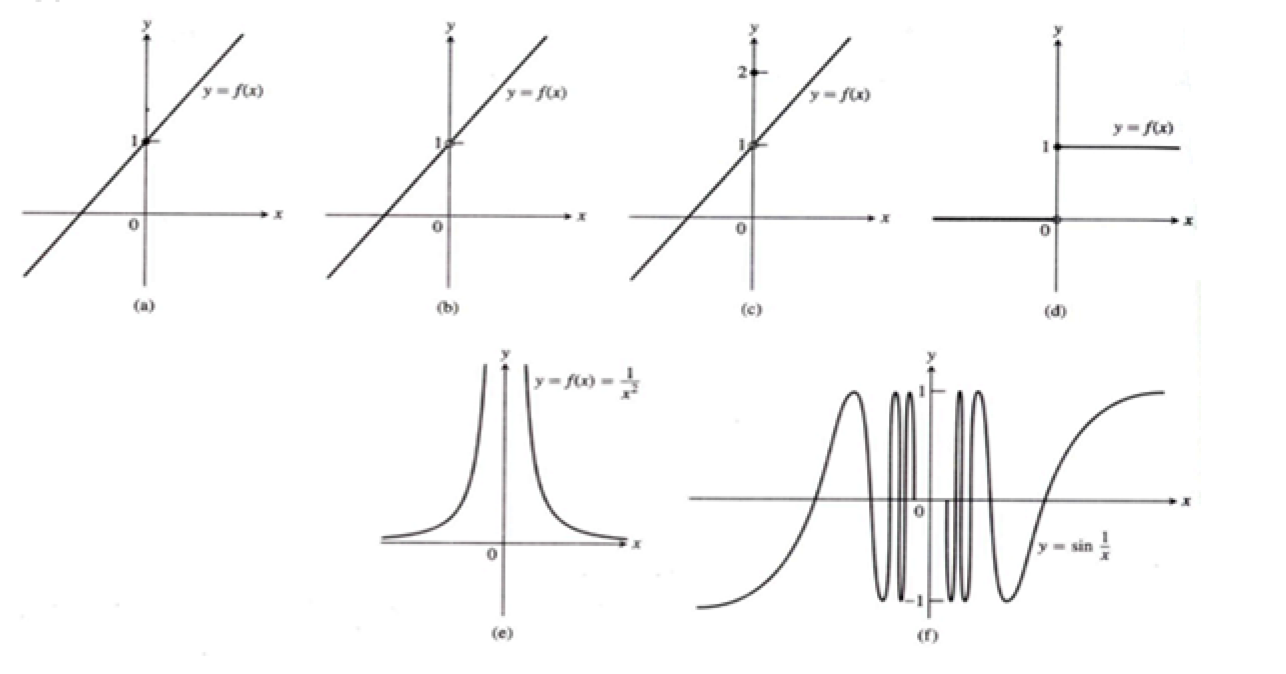
Thus, a function is continuous at a left endpoint a of its domain if it is right-continuous at a and continuous at a right endpoint b of its domain if it is left-continuous at b.

A function is continuous at an interior point c of its domain if and only if it is both right-continuous and left-continuous at c.



If a function f is not continuous at a point c, then we say that f is ***discontinuous*** at c and c is called a ***point of discontinuity*** of f.

Types of discontinuities:



The function in (a) is continuous at $x = 0$.

The function in (b) would be continuous if it had $f(0) = 1$.

The function in (c) would be continuous if $f(0)$ were 1 instead of 2.

The discontinuities in (b) and (c) are ***removable***. Each function has a limit as $x \to 0$, and we can remove the discontinuity by setting $f(0)$ equal to this limit.

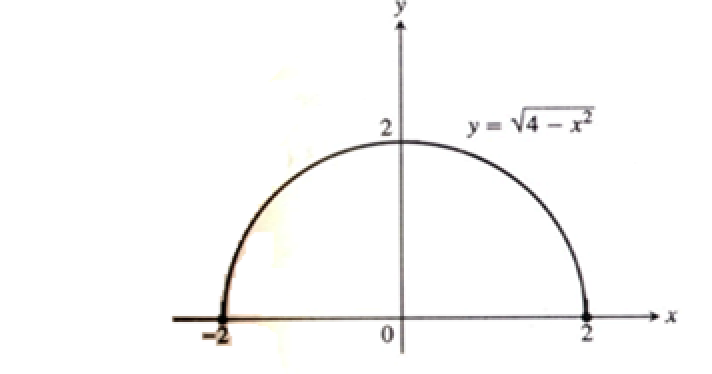
The discontinuities in parts (d) to (f) are of different nature: $\lim_{x \to 0} f(x)$ does not exist.

The step function in (d) has a **jump discontinuity**: the one-sided limits exist but have different values.

The function $f(x) = \frac{1}{x^2}$ in (e) has an **infinite discontinuity.** These discontinuities are the ones most frequently encountered in applications. The function in (f) has an **oscillating discontinuity** at the origin because it oscillates too much to have limit as $x \to 0$.

Example 1*: A function continuous through out its domain.*

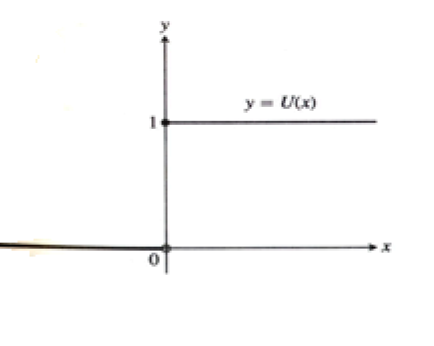
The function $f(x) = \sqrt{4 - x^2}$ is continuous at every point of its domain, $[-2, 2]$.



This includes $x = -2$, where f is right-continuous, and $x = 2$, where f is left-continuous.

Example 2*: The unit step function has jump discontinuity.*

The unit step function is graphed below.



It is right-continuous at $x = 0$, but is neither left-continuous there nor continuous at $x = 0$. It has a jump discontinuity at $x = 0$.

Continuity Test

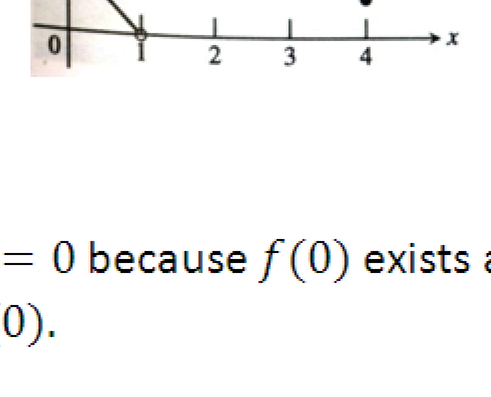
A function $f(x)$ is continuous at $x = c$ if and only if it meets the following three conditions.

1. $f(c)$ exists (c lies in the domain of f)
2. $\lim_{x \to c} f(x)$ exists (f has a limit as $x \to c$)
3. $\lim_{x \to c} f(x) = f(c)$ (the limit equals the function value)

For one-sided continuity and continuity at an endpoint, the limits in parts 2 and 3 of the test should be replaced by the appropriate one-sided limits.

Example 3:

Consider the function $y = f(x)$, in the figure below, whose domain is the closed interval $[0, 4]$. Discuss the continuity of f at $x = 0, 1, 2, 3, 4$.



Solution:

f is continuous at $x = 0$ because $f(0)$ exists and $\lim_{x \to 0^+} f(x) = 1 = f(0)$.

f is discontinuous at $x = 1$ because $\lim_{x \to 1} f(x)$ does not exist; f has different right- and left- hand limits at the interior point $x = 1$. However, f is right continuous at $x = 1$ because $f(1)$ exists, $\lim_{x \to 1^+} f(x) = 1$, and this equals the function value.

Note that $\lim_{x \to 1^+} f(x) = 1$, $\lim_{x \to 1^-} f(x) = 0$. There fore $x = 1$ is a point of discontinuity and it is a jump discontinuity

f is discontinuous at $x = 2$ because $\lim_{x \to 2} f(x) \neq f(2)$.

Therefore $x = 2$ is a removable discontinuity, by setting $\lim_{x \to 2} f(x) = 1$.

f is continuous at $x = 3$ because $f(3)$ exists, $\lim_{x \to 3} f(x) = 2$, and this is equal to the function value.

f is discontinuous at the right endpoint $x = 4$ because $\lim_{x \to 4^-} f(x) \neq f(4)$.

6.2

Rules of Continuity

Learning objectives:

- To state the properties of continuous functions.
- To study the continuity of polynomials, rational functions, absolute value function and trigonometric functions.
- To define the continuous extension of a function to a point.
  And
- To practice related problems.

Rules of Continuity

Algebraic combinations of continuous functions are continuous wherever they are defined

Theorem: Continuity of Algebraic Combinations

If functions f and g are continuous at $x = c$, then the following functions are continuous at $x = c$:

1. $f + g$ and $f - g$
2. fg
3. kf, where k is any number
4. $\frac{f}{g}$, provided $g(c) \neq 0$
5. $(f(x))^{\frac{m}{n}}$, m and n are integers, $n \neq 0$.

As a consequence, polynomials and rational functions are continuous at every point where they are defined.

Theorem: Continuity of Polynomials and Rational Functions

Every polynomial is continuous at every point of the real line. Every rational function is continuous at every point where its denominator is different from zero.

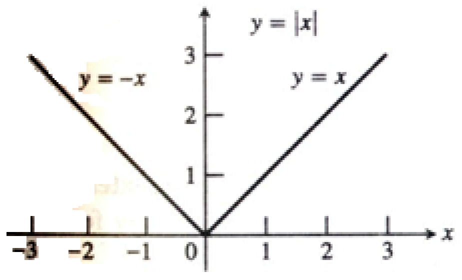
Example 1: The functions $f(x) = x^4 + 20$ and $g(x) = 5x(x-2)$ are continuous at every value of x. The function

$$r(x) = \frac{x^4+20}{5x(x-2)}$$

is continuous at every value of x except $x = 0$ and $x = 2$, where the denominator is 0.

Example 2: Continuity of $f(x) = |x|$

The function $f(x) = |x|$ is continuous at every value of x.



If $x > 0$, we have $f(x) = x$ is a polynomial.

If $x < 0$, we have $f(x) = -x$ is another polynomial. Finally, at the origin, $\lim_{x\to 0}|x| = 0 = |0|$.

Example 3:

We will later show that the functions $\sin x$ and $\cos x$ are continuous at every value of x. It then follows that the quotients

$$\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{\cos x}{\sin x}$$

$$\sec x = \frac{1}{\cos x}, \quad \csc x = \frac{1}{\sin x}$$

are continuous at every point where they are defined.

6.3
Continuity on Intervals

Learning objectives:

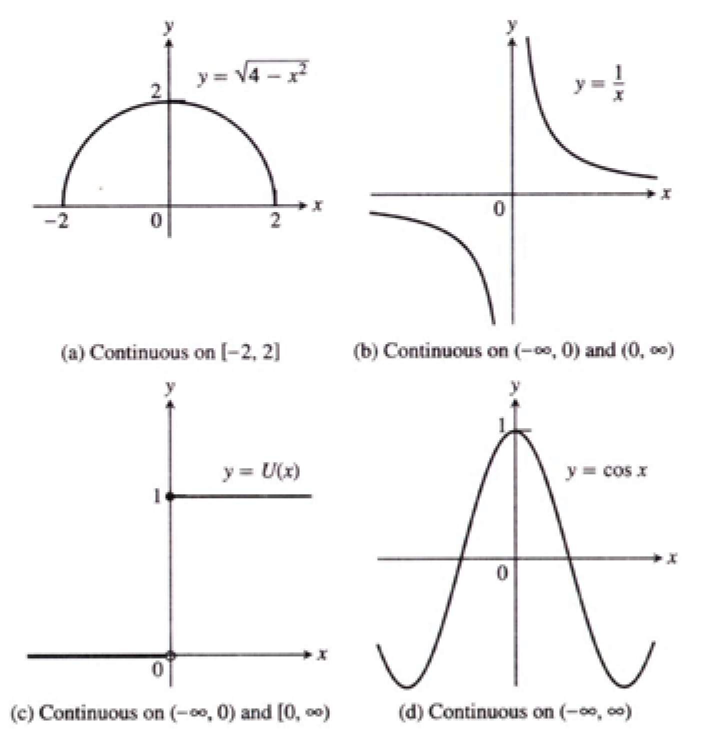
- To define continuity of a function on its domain.
- To study intermediate value theorem and its application to assert the existence of a zero of a function.
  And
- To practice the related problems.

A function is called **continuous** if it is continuous everywhere in its domain.
A function that is not continuous throughout its entire domain may be continuous when restricted to particular intervals within the domain.

A function f is said to be ***continuous on an interval*** I in its domain if $\lim_{x\to c} f(x) = f(c)$ at every interior point c and if the appropriate one-sided limits equal the function values at the endpoints.
A function continuous on an interval I is automatically continuous on any interval contained in I.

Polynomials are continuous on every interval, and rational functions are continuous on every interval on which they are defined.

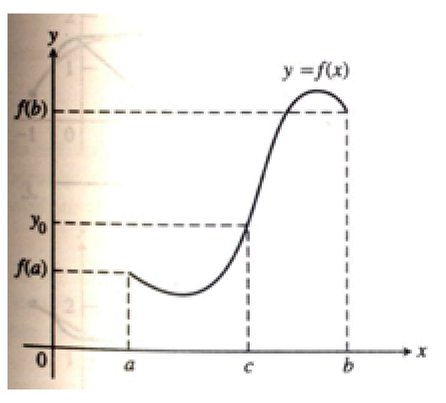
Example 1: Functions continuous on intervals



Functions that are continuous on intervals have properties that make them particularly useful in applications. One of these is the intermediate value property.
A function is said to have the ***intermediate value property*** if whenever it takes on two values, it also takes on all the values in between.

Theorem: The Intermediate Value Theorem

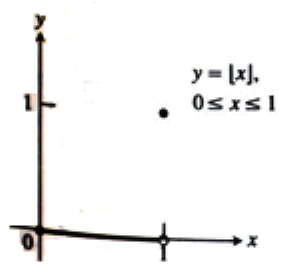
Suppose $f(x)$ is continuous on an interval I, and a and b are any two points of I. Then if y_0 is a number between $f(a)$ and $f(b)$, there exists a number c between a and b such that $f(c) = y_0$.



The function f, being continuous on $[a, b]$, takes on every value between $f(a)$and $f(b)$.

The proof of the Intermediate Value Theorem depends on the completeness property of the real number system.

The continuity of f on I is essential to the theorem. If f is discontinuous even at one point of f, the theorem does not apply. For example, it will not apply for the function graphed below.



The function $f(x) = [x], 0 \le x \le 1$, does not take on any value between $f(0) = 0$ and $f(1) = 1$.

The above theorem is the reason for the graph of a function continuous on an interval I cannot have any breaks. It will be connected, a single, unbroken curve, like the graph of $\sin x$. It will not have jumps like the graph of the greatest integer function $[x]$ or separate branches like the graph of $\frac{1}{x}$.

We call a solution of the equation $f(x) = 0$ a **root** or **zero** of the function f. The Intermediate Value Theorem tells the following:
If f is continuous, then any interval on which f changes sign must contain a zero of the function.

Example 2: Is any real number exactly 1 less than its cube?

Solution: Any such number must satisfy the equation
$$x = x^3 - 1$$
$$i.e.,\ x^3 - x - 1 = 0$$

Hence we are looking for zeros of $f(x) = x^3 - x - 1$. By trial, we find that $f(1) = -1$ and $f(2) = 5$. Then, by the Intermediate Value Theorem, there is at least one number in [1,2] where f is zero. The answer to the question is then "yes".

6.4

Tangent Lines

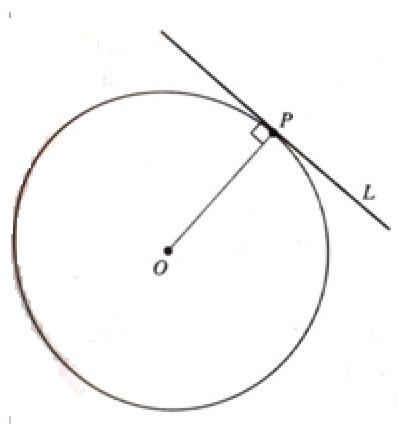
Learning objectives:

- To define the tangent to a curve at a point on the curve and to find it.
  And
- To practice the related problems.

Tangent Lines

Tangent to a Curve

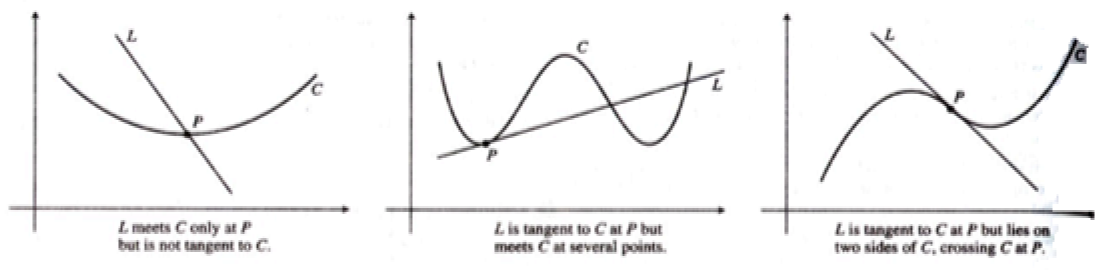
From the geometry, we know the tangents to circles. A line L is tangent to a circle at a point P if L passes through P and is perpendicular to the radius at P. Such a line just *touches* the circle.



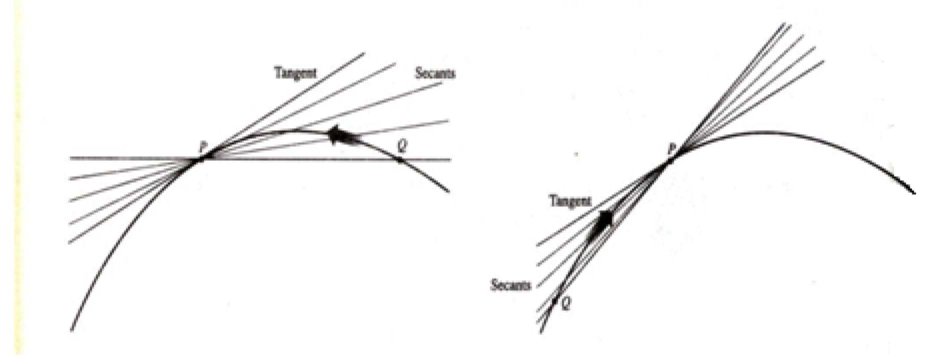
The following statements are valid.

1. L passes through P and is perpendicular to the line from P to the center of C.
2. L passes through only one point of C, namely P.
3. L passes through P and lies on one side of C only.

These statements may not apply consistently for more general curves. Most curves do not have centers, and a line we may want to call tangent may intersect C at other points or cross C at the point of tangency.



To define tangency for general curves, we take into account the behavior of the secants through P and nearby points Q (on C) as Q moves toward P along the curve.



The procedure is as follows:

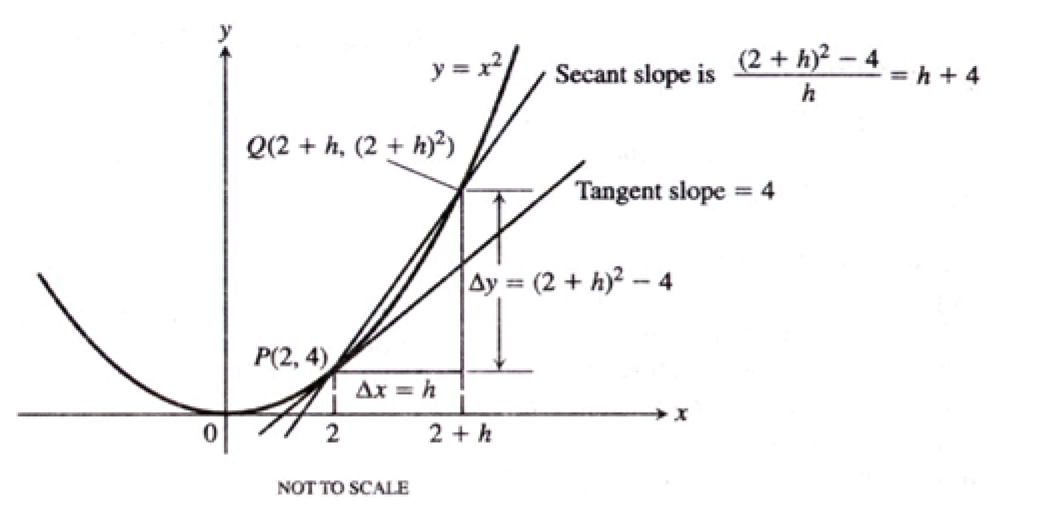
1. We calculate the slope of the secant PQ
2. Investigate the limit of the secant slope as Q approaches P along the curve.
3. If the limit exists, we take it to be the slope of the curve at P and define the tangent to the curve at P to be the line through P with this slope.

Example 1: Find the slope of the parabola $y = x^2$ at the point P(2,4). Write an equation for the tangent to the parabola at this point.

Solution: Consider the secant line through $P(2,4)$ and Q$(2+h, (2+h)^2)$ nearby.

Secant slope $= \frac{\Delta y}{\Delta x} = \frac{(2+h)^2-2^2}{h} = \frac{h^2+4h+4-4}{h} = \frac{h^2+4h}{h} = h+4$

If $h > 0$, Q lies above and to the right of P, as in the figure below.



As Q approaches P along the curve, *h* approaches zero and the secant slope approaches 4:

$$\lim_{h\to 0} h + 4 = 4$$

We take 4 to be the parabola's slope at P. The tangent to the parabola at P is the line through P with slope 4. The equation of the tangent to the parabola at P is,

$y = 4 + 4(x-2)$ Point-slope equation

$\Rightarrow y = 4x - 4$