## 7.1

## The Derivative of a Function

### Leaning objectives:

In this module, we study

- To define the derivative of a function with respect to its independent variable.
- To compute the derivatives using the definition.
- To define differentiability on an interval.

  And
- To solve the problems related to the above concepts.

### The Derivative of a Function

In the unit on limits, we defined the slope of a curve $y = f(x)$ at the point where $x = x_0$ to be

$$m = \lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h}$$

We called this limit the derivative of $f$ at $x_0$. Now, we investigate the derivative as a function derived from $f$ by considering the limit of the difference quotient at each point of the domain.

**Definition:**

*The* ***derivative*** *of the function $f$ with respect to the variable $x$ is the function $f'$ whose value at $x$ is*

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

*provided the limit exists.*

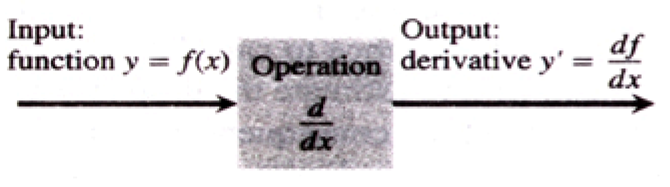
*If $f'(x)$ exists, we say that $f$* ***has a derivative*** *at $x$ or $f$* ***is differentiable*** *at $x$.*

The domain of $f'$, the set of points in the domain of $f$ for which the limit exists. may be smaller than the domain of $f$.

There are many ways to denote the derivative of a function $y = f(x)$, where $x$ is the independent variable and $y$ is the dependent variable.

Besides $f'$, the other notations are $y', \frac{dy}{dx}, \frac{df}{dx}, \frac{d}{dx}f(x), D_x f$.

These are pronounced as "$f$ prime", "$y$ prime", "$dy$ by $dx$", "$df$ by $dx$", "$d$ by $dx$ of $f(x)$" and "$dx$ of $f$" respectively. The notation $\dot{y}$ ($y$ dot) is used for time derivatives. The notation $\frac{dy}{dx}$ is also read as "the derivative of $y$ with respect to $x$," and $\frac{df}{dx}$ and $\frac{d}{dx}f(x)$ as "the derivative of $f$ with respect to $x$."




The above figure gives a flow diagram for the operation of taking a derivative with respect to $x$.

### Calculating the Derivatives from the Definition

The process of calculating a derivative is called **differentiation**. The procedure for calculating the derivative is as follows:

1. Write expressions for $f(x)$ and $f(x+h)$
2. Expand and simplify the difference quotient $\frac{f(x+h)-f(x)}{h}$
3. Using the simplified quotient, find $f'(x)$ by evaluating the limit

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

**Example 1:** The derivative of $y = mx + b$ is $m$ at any $x$.

**Solution:**

We have $y = f(x)$ where $f(x) = mx + b$.

Now, $\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

$$= \lim_{h \to 0} \frac{m(x+h)+b-mx-b}{h} = \lim_{h \to 0} \frac{mh}{h} = m$$

**Example 2:**

a) Differentiate $f(x) = \frac{x}{x-1}$

b) Where does the curve $y = f(x)$ have slope $-1$?

**Solution:**

a)

$$\frac{f(x+h)-f(x)}{h} = \frac{\frac{x+h}{x+h-1} - \frac{x}{x-1}}{h}$$

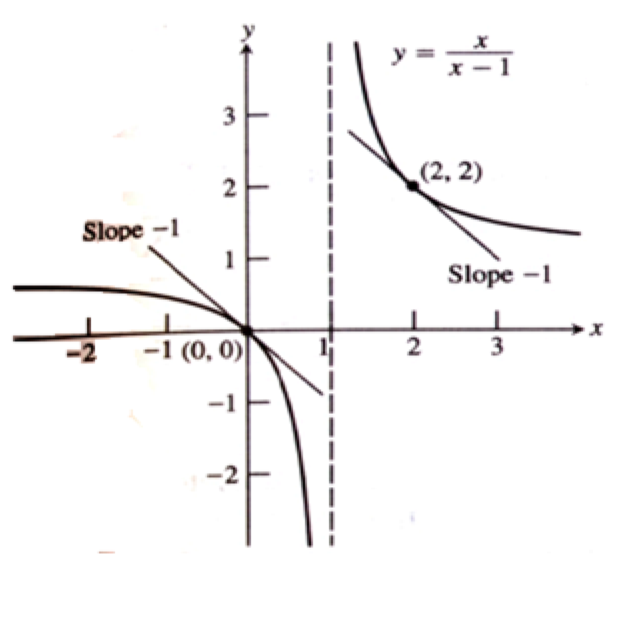
$$= \frac{1}{h} \cdot \frac{(x+h)(x-1)-x(x+h-1)}{(x+h-1)(x-1)}$$

$$= \frac{1}{h} \cdot \frac{-h}{(x+h-1)(x-1)}$$

$$f'(x) = \lim_{h \to 0} \frac{-h}{(x+h-1)(x-1)} = \frac{-1}{(x-1)^2}$$

b) The slope of the curve $y = f(x)$ is $f'(x)$. Given that the slope is $-1$

$$\frac{-1}{(x-1)^2} = -1 \implies (x-1)^2 = 1 \implies x = 2 \text{ or } x = 0$$




**Example 3:**

a) Find the derivative of $y = \sqrt{x}$ for $x > 0$.

b) Find the tangent line to the curve $y = \sqrt{x}$ at $x = 4$.

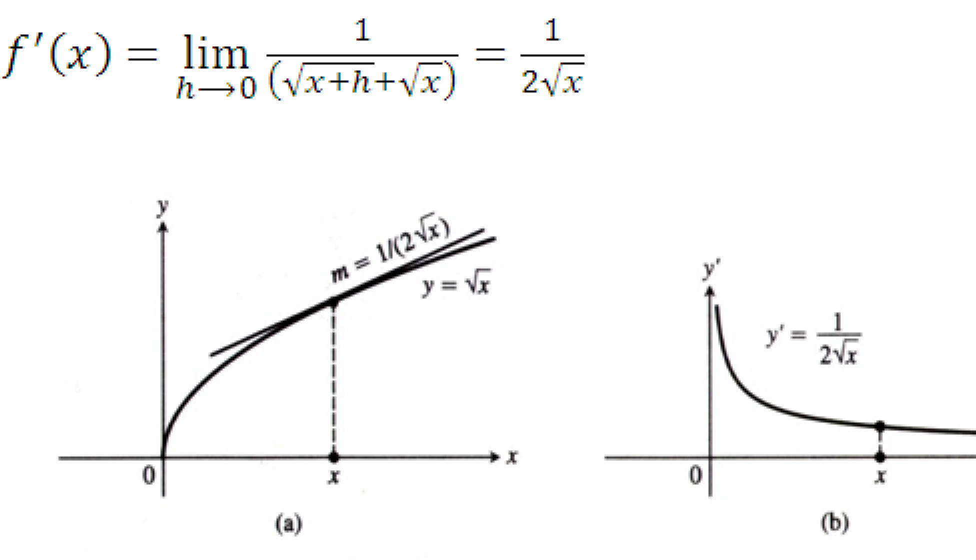
***Solution:***

a)

$$\frac{f(x+h)-f(x)}{h} = \frac{\sqrt{x+h}-\sqrt{x}}{h} \quad \text{Multiply by } \frac{(\sqrt{x+h}+\sqrt{x})}{(\sqrt{x+h}+\sqrt{x})}$$

$$= \frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})} = \frac{1}{(\sqrt{x+h}+\sqrt{x})}$$

$$f'(x) = \lim_{h \to 0} \frac{1}{(\sqrt{x+h}+\sqrt{x})} = \frac{1}{2\sqrt{x}}$$




The function is defined at $x = 0$, but its derivative is not.

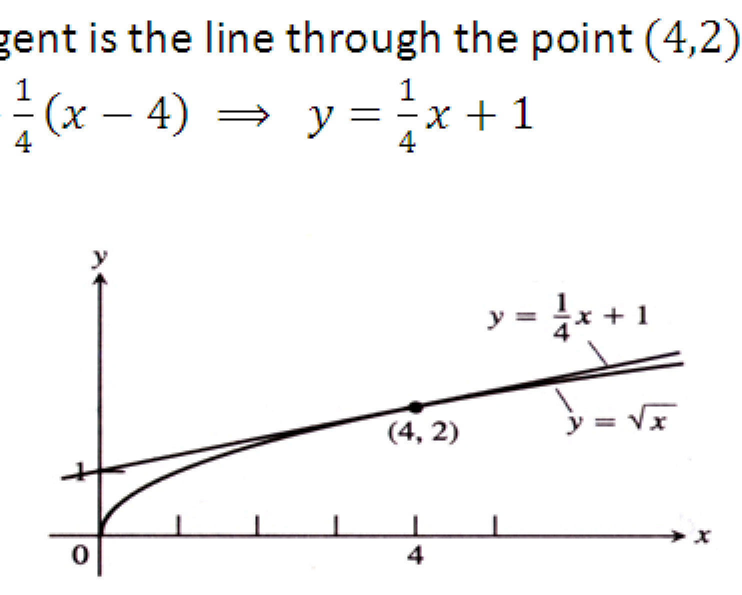
b)

The slope of the curve at $x = 4$ is

$$\frac{dy}{dx}\Big|_{x=4} = \frac{1}{2\sqrt{x}}\Big|_{x=4} = \frac{1}{4}$$

The tangent is the line through the point $(4,2)$ with slope $1/4$.

$$y = 2 + \frac{1}{4}(x-4) \implies y = \frac{1}{4}x + 1$$




The value $f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$ of the derivative of $y = f(x)$ with respect to $x$ at $x = a$ can be denoted in the following ways:

$$y'|_{x=a} = \frac{dy}{dx}\Big|_{x=a} = \frac{d}{dx}f(x)\Big|_{x=a}$$

Here, the symbol $|_{x=a}$, called an **evaluation symbol**, tells us to evaluate the expression to its left at $x = a$.

## 7.2
## Derivatives and Continuity

### Learning objectives:

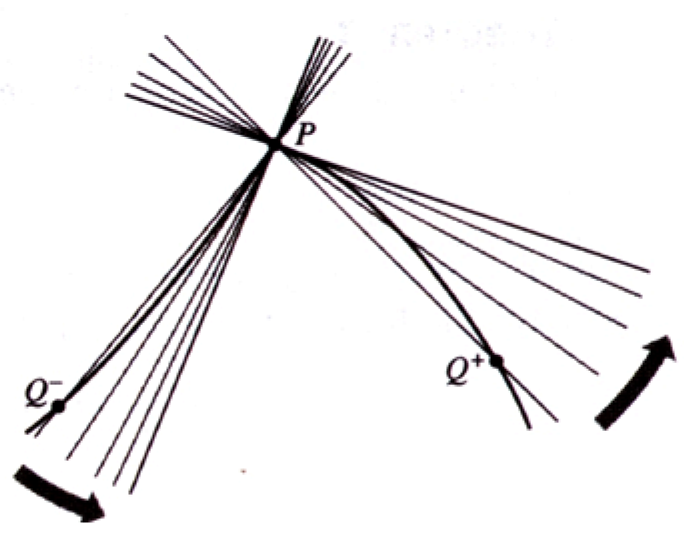
- To investigate the reasons for a function fail to have a derivative at a point.
- To prove that a function is continuous at every point where it has a derivative.
  And
- To practice related problems.

**Derivatives and Continuity**

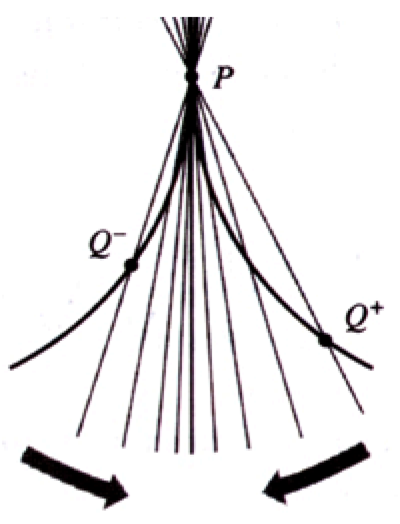
**Function Not Having a Derivative at a Point**

A function has a derivative at a point $x_0$ if the slopes of secant lines, through $P(x_0, f(x_0))$ and a nearby point $Q$ on the graph, approach a limit as $Q$ approaches $P$. Whenever the secants fail to take up a limiting position or become vertical as $Q$ approaches $P$, the derivative does not exist. A function whose graph is otherwise smooth will fail to have a derivative at a point where the graph has

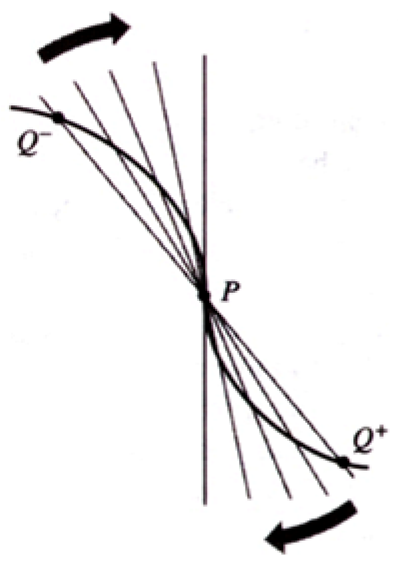
1. A ***corner***, where the one-sided derivatives differ



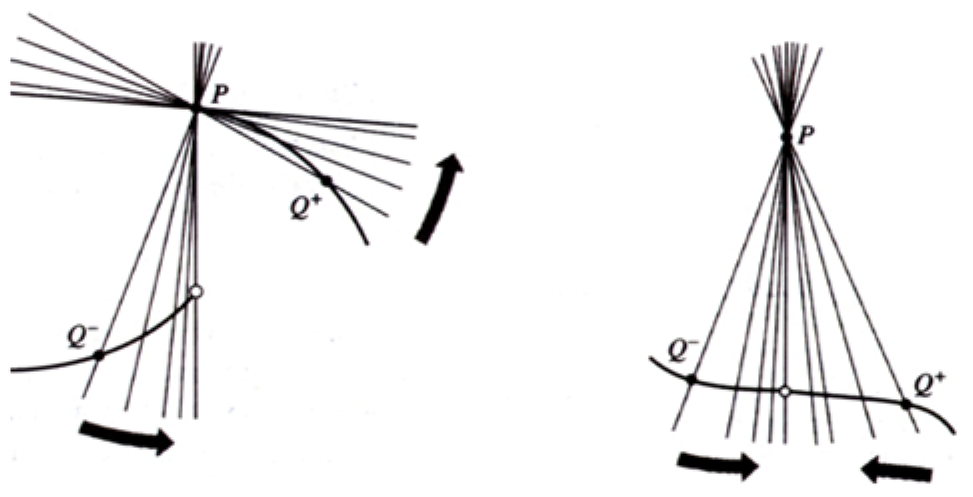

2. A ***cusp***, where the slope of $PQ$ approaches $\infty$ from one side and $-\infty$ from the other




3. A ***vertical tangent***, where the slope of $PQ$ approaches $\infty$ from both sides or approaches $-\infty$ from both sides




4. A ***discontinuity***.




**Differentiability and Continuity**

*A function is continuous at every point where it has a derivative.*

**Theorem:**

If $f$ has a derivative at $x = c$ , then $f$ is continuous at $x = c$.

**Proof:**

Given that $f'(c)$ exists ,we must show that $\lim_{h\to 0} f(x) = f(c)$, or equivalently, that $\lim_{h\to 0} f(c+h) = f(c)$. If $h \neq 0$, then

$$f(c+h) - f(c) = \frac{f(c+h)-f(c)}{h}.h$$

$$i.e.,\quad f(c+h) = f(c) + \frac{f(c+h)-f(c)}{h}.h$$

Now, we take limits as $h \to 0$.

$$\lim_{h\to 0} f(c+h) = \lim_{h\to 0} f(c) + \lim_{h\to 0} \frac{f(c+h)-f(c)}{h}.\ \lim_{h\to 0} h$$

$$= f(c) + f'(c).0 = f(c)$$

Similar arguments with one-sided limits show that *if $f$ has a derivative from one side (right or left) at $x = c$ , then $f$ is continuous from that side at $x = c$.*

The above theorem says that

*If a function has a discontinuity at a point (for instance, a jump discontinuity) then it can not be differentiable there.*

**Example 1:** The greatest integer function $y = \lfloor x \rfloor = int\ x$ is not differentiable at every integer $x = n$ (since it jumps at every integer).

The converse of the above theorem is false. A function need not have a derivative at a point where it is continuous.

**Example 2:** The absolute value function $y = |x|$ is continuous at $x = 0$ is but it is not differentiable at $x = 0$.

## 7.3

## Differentiation Rules – Sums and Differences

### Learning objectives:

- To formulate $(i)$ the power rule for positive integers, $(ii)$ the constant multiple rule, $(iii)$ the sum rule and $(iv)$ the difference rule for differentiation.
  And
- To practice the related problems.

### Differentiation Rules – Sums and Differences

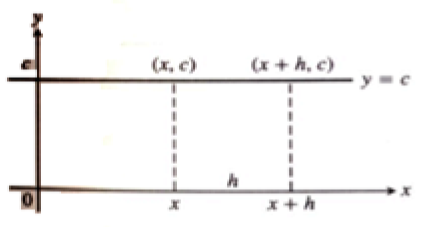
In this module and the next, we formulate some rules for differentiation of functions without having to apply the definition each time.
The first rule of differentiation is that *the derivative of every constant function is zero.*

**Rule 1: Derivative of a Constant**

If $c$ is a constant, then $\frac{d}{dx}(c) = 0$

**Proof:**

We apply the definition of derivative to $f(x) = c$, the function whose outputs have the constant value $c$.




At every value of $x$, we find that

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{c-c}{h} = 0$$

**Example 1:** $\frac{d}{dx}(8) = 0$ , $\frac{d}{dx}\left(-\frac{1}{2}\right) = 0$ , $\frac{d}{dx}(\sqrt{3}) = 0$

The next rule tells how to differentiate $x^n$ , if $n$ is a positive integer.

**Rule 2: Power Rule for Positive Integers**

If $n$ is a positive integer, then

$$\frac{d}{dx}x^n = nx^{n-1}$$

**Proof:**

Let $f(x) = x^n$.Since $n$ is a positive integer, we use the fact that

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})$$

Now,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h\to 0}\frac{h[(x+h)^{n-1}+(x+h)^{n-2}x+\cdots+(x+h)(x)^{n-2}+(x)^{n-1}]}{h}$$

$$= \lim_{h\to 0}[(x+h)^{n-1} + (x+h)^{n-2}x + \cdots + (x+h)x^{n-2} + x^{n-1}]$$

$$= nx^{n-1}$$

**Example 2:** $\frac{d}{dx}x = 1$ , $\frac{d}{dx}x^2 = 2x$ , $\frac{d}{dx}x^3 = 3x^2$

The next rule says that *when a differentiable function is multiplied by a constant, its derivative is multiplied by the same constant.*

**Rule 3: The Constant Multiple Rule**

If $u$ is a differentiable function of $x$, and $c$ is a constant, then

$$\frac{d}{dx}(c\,u) = c\,\frac{du}{dx}$$

**Proof:**

$$\frac{d}{dx}(cu) = \lim_{h\to 0}\frac{c\,u(x+h)-c\,u(x)}{h}$$

$$= c\lim_{h\to 0}\frac{u(x+h)-u(x)}{h}$$

$$= c\frac{du}{dx}$$

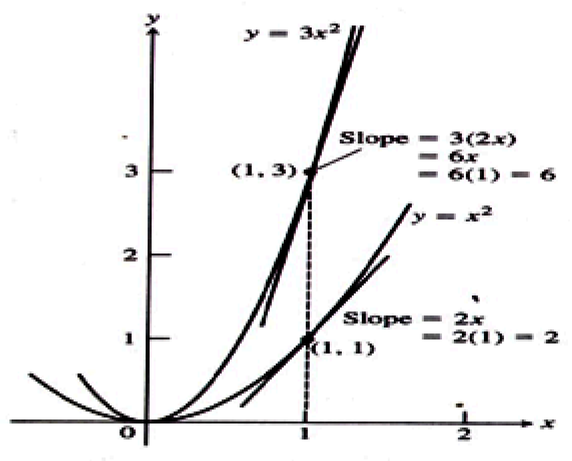
In particular, if $n$ is a positive integer, then

$$\frac{d}{dx}(cx^n) = cnx^{n-1}$$

**Example 3**: The derivative formula

$$\frac{d}{dx}(3x^2) = 3\times 2x = 6x$$

says that if we rescale the graph of $y = x^2$ by multiplying each $y$-coordinate by 3, then we multiply the slope at each point by 3.




**A special case:**

*The derivative of the negative of a differentiable function is the negative of the function's derivative.*

$$\frac{d}{dx}(-u) = \frac{d}{dx}(-1.u) = -1.\frac{d}{dx}(u) = -\frac{du}{dx}$$

The next rule says that *the derivative of the sum of two differentiable functions is the sum of their derivatives.*

## 7.4

## Differentiation Rules – Products and Quotients

### Learning objectives:

- To derive the following rules of differentiation:
  - ❖ The product rule.
  - ❖ The quotient rule.
  - ❖ Power Rule for negative Integers.

  And
- To practice related problems.

### Differentiation Rules – Products and Quotients

**Products and Quotients**

While the derivative of the sum of two functions is the sum of their derivatives, the derivative of the product of two functions is *not the product of their derivatives*. For instance,

$$\frac{d}{dx}(x\cdot x)=\frac{d}{dx}(x^2)=2x, \qquad \text{while } \frac{d}{dx}(x)\cdot\frac{d}{dx}(x)=1\cdot 1=1$$

The derivative of a product of two functions is the sum of *two* products, as we see now from the next rule.

**Rule 5: The Product Rule**

If $u$ and $v$ are differentiable at $x$, then so is their product $uv$, and

$$\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$$

The derivative of the product $uv$ is $u$ times the derivative of $v$ plus $v$ times the derivative of $u$. In prime notation,

$$(uv)' = uv' + vu'.$$

**Proof:**

$$\begin{aligned}\frac{d}{dx}(uv) &= \lim_{h\to 0}\frac{u(x+h)v(x+h)-u(x)v(x)}{h}\\ &= \lim_{h\to 0}\frac{u(x+h)v(x+h)-u(x+h)v(x)+u(x+h)v(x)-u(x)v(x)}{h}\\ &= \lim_{h\to 0}\left[u(x+h)\frac{v(x+h)-v(x)}{h}+v(x)\frac{u(x+h)-u(x)}{h}\right]\\ &= \lim_{h\to 0}u(x+h)\cdot\lim_{h\to 0}\frac{v(x+h)-v(x)}{h}+v(x)\cdot\lim_{h\to 0}\frac{u(x+h)-u(x)}{h}\\ &= u\frac{dv}{dx}+v\frac{du}{dx}\end{aligned}$$

**Example 1: Find the derivative of** $y = (x^2+1)(x^3+3)$.

**Solution:**

From the product rule with $u=x^2+1$ and $v=x^2+3$, we find

$$\begin{aligned}\frac{d}{dx}\left[(x^2+1)(x^3+3)\right] &= (x^2+1)\frac{d}{dx}(x^3+3)+(x^3+3)\frac{d}{dx}(x^2+1)\\ &= (x^2+1)(3x^2)+(x^3+3)(2x)\\ &= 3x^4+3x^2+2x^4+6x = 5x^4+3x^2+6x\end{aligned}$$

The above example can be done as well (perhaps better) by multiplying out the original expression for $y$ and differentiating the resulting polynomial.

$$y=(x^2+1)(x^3+3)=x^5+x^3+3x^2+3$$

$$\frac{dy}{dx}=5x^4+3x^2+6x$$

There are times, however, when the product rule must be used.

## 7.5

# Second and Higher Order Derivatives

### Learning objectives:

- To find the second and higher order derivatives of a function $f(x)$.
  And
- To practice the related problems.

### Second and Higher Order Derivatives

The derivative $y' = \frac{dy}{dx}$ is the ***first (first order) derivative*** of $y$ with respect to $x$. This derivative may itself be a differentiable function of $x$; if so, its derivative

$$y'' = \frac{dy'}{dx} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

is called the ***second (second order) derivative*** of $y$ with respect to $x$. The symbol $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ does not mean multiplication. It means *the derivative of the derivative.*

If $y''$ is differentiable, its derivative $y''' = \frac{dy''}{dx} = \frac{d^3y}{dx^3}$ is the ***third (third order) derivative*** of $y$ with respect to $x$. The names continue with

$$y^{(n)} = \frac{d}{dx}y^{(n-1)}$$

denoting ***the $n^{th}$ ($n^{th}$ order) derivative*** of $y$ with respect to $x$, for any positive integer $x$.

We can interpret the second derivative as the rate of change of the slope of the tangent to the graph of $y = f(x)$ at each point.

**Example 1: The first four derivatives of $y = x^3 - 3x^2 + 2$ are**

First Derivative: $y' = 3x^2 - 6x$

Second Derivative: $y'' = 6x - 6$

Third Derivative: $y''' = 6$

Fourth Derivative: $y^{(4)} = 0$

The function has derivatives of all orders, the fifth and later derivatives all being zero.

7.6

# Rates of Change

**Learning objectives:**

- To define the instantaneous rate of change of a function.
- To discuss the motion of an object moving along a coordinate line.
- To discuss the free fall of the falling bodies.
  And
- To practice the related problems.

## Rates of Change

The ***average rate of change*** of a function $f(x)$ with respect to $x$ over the interval from $x_0$ to $x_0 + h$ is $\frac{f(x_0+h)-f(x_0)}{h}$. The (***instantaneous***) rate of change of $f$ with respect to $x$ at $x_0$ is the derivative

$$f'(x_0) = \lim_{x \to 2} \frac{f(x_0+h)-f(x_0)}{h}$$

provided the limit exists.
Thus, instantaneous rates are limits of average rates.

**Example 1:**

The area $A$ of a circle is related to its diameter $D$ by the equation $A = \frac{\pi}{4} D^2$. How fast is the area changing with respect to the diameter when the diameter is $10\ m$?

**Solution:**

The rate of change of the area with respect to the diameter is

$$\frac{dA}{dD} = \frac{\pi}{4}(2D) = \frac{\pi D}{2}$$

When $D = 10\ m$, the area is changing at the rate

$$\left(\frac{\pi}{2}\right) 10 = 5\pi\ \ m^2.$$

## 7.7

## Sensitivity to Change

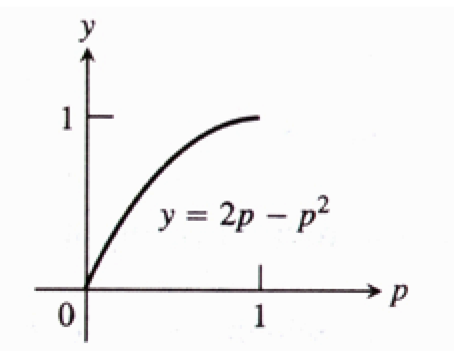
### Learning objectives:

- To study the sensitivity of a function $f(x)$ to changes in $x$.
- To study the concept of *marginal* of a function $f(x)$ in economics as the rate of change of $f(x)$ w.r.t $x$.
  And
- To practice the related problems.

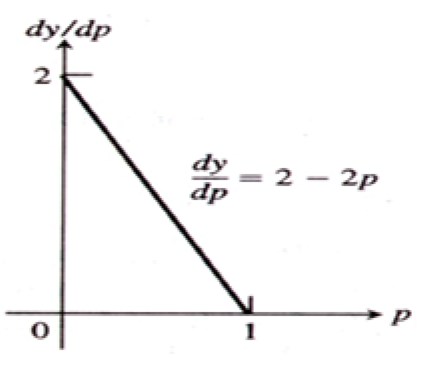
### Sensitivity to Change

When a small change in $x$ produces a large change in the value of a function $f(x)$, we say that the function is relatively ***sensitive*** to changes in $x$. The derivative $f'(x)$ is a measure of the sensitivity to change at $x$.

**Example 1:**
If $p$ (a number between 0 and 1) is the frequency of the gene for smooth skin in peas and $(1-p)$ is the frequency of the gene for wrinkled skin in peas, experimental records show that the proportion of smooth-skinned peas in the population at large is $y = 2p(1-p) + p^2 = 2p - p^2$




The graph of $y$ versus $p$ suggests that the value of $y$ is more sensitive to a change in $p$ when $p$ is small than when $p$ is large.



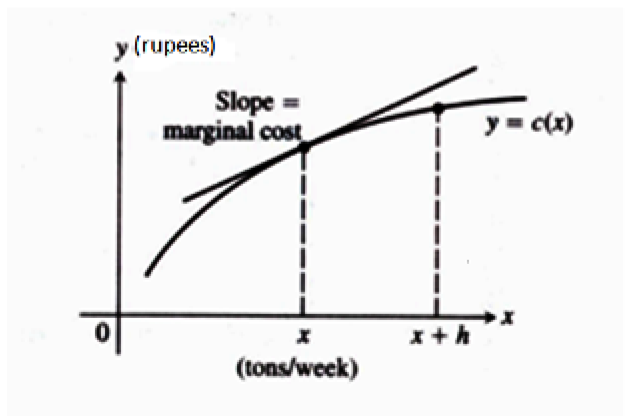

This is evident form of the derivative graph, which shows that $dy/dp$ is close to 2 when $p$ is near zero and close to zero when $p$ is near 1.

Economists call rates of change and derivatives as ***marginals***. In a manufacturing operation, the cost of production $c(x)$ is a function of $x$, the number of units produced. The *marginal cost of production* is the rate of change of cost $c$ with respect to the level of production $x$ , so it is $dc/dx$.

For example, let $c(x)$ represent the rupees needed to produce $x$ tons of steel in one week. It costs more to produce $x+h$ units per week and the cost difference divided by $h$ is the average cost of producing each additional ton and it is given by : $\frac{c(x+h)-c(x)}{h}$. The limit of this ratio as $h \to 0$ is the ***marginal cost*** of producing more steel when the current production is $x$ tons:

$$\frac{dc}{dx} = \lim_{h\to 0} \frac{c(x+h)-c(x)}{h}$$

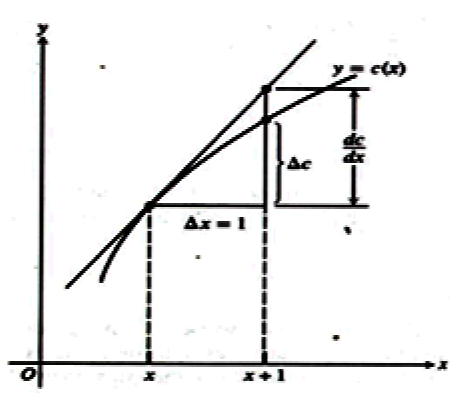
is the marginal cost of production.




Sometimes, the marginal cost of production is loosely defined to be the extra cost of producing one unit:

$$\frac{\Delta c}{\Delta x} = \frac{c(x+1)-c(x)}{1}$$

which is approximately the value of $dc/dx$ at $x$. The approximation works best for large values of $x$.




**Example 2:**
Suppose it costs $c(x) = x^3 - 6x^2 + 15x$ rupees to produce $x$ radiators when 8 to 30 radiators are produced. Your shop currently produces 10 radiators a day. About how much extra will it cost to produce one more radiator a day?

**Solution:**
The cost of producing one more radiator a day when 10 are produced is *about* $c'(10)$:

$$c'(x) = \frac{d}{dx}(x^3 - 6x^2 + 15x\ ) = 3x^2 - 12x + 15$$

$$c'(10) = 3(10)^2 - 12(10) + 15 = 195$$

The additional cost is about $Rs$ 195.

**Example-Marginal tax rate:**
If your marginal income tax rate is 28% and your income increases by 1000, you can expect to have to pay an extra 280 in income taxes. This means that at your current income level $I$, the rate of increase of taxes $T$ with respect to income is $dT/dI = 0.28$ . You will pay $Rs$ 0.28 out of every extra rupee you earn in taxes.

**Example 3:**
If $r(x) = x^3 - 3x^2 + 12x$ gives the revenue in rupees from selling $x$ thousand candy bars a week, $5 \le x \le 20$, the marginal revenue when $x$ thousand are sold is

$$r'(x) = \frac{d}{dx}(x^3 - 3x^2 + 12x) = 3x^2 - 6x + 12$$

As with marginal cost, the marginal revenue function estimates the increase in revenue that will result from selling one additional unit. If you currently sell 10 thousand candy bars a week, you can expect your revenue to increase by about

$$r'(10) = 3(10)^2 - 6(10) + 12 = 252$$

if you increase sales to 11 thousand bars a week.

## 7.8

## Some Special Limits

### Learning objectives:

- To prove $-|\theta| < \sin\theta < |\theta|$, $-|\theta| < 1-\cos\theta < |\theta|$ and $\lim\limits_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$, when $\theta$ is measured in radians.
  And
- To practice the related problems.

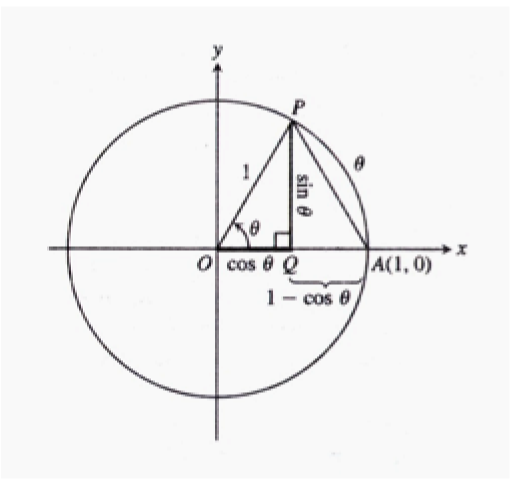
### Some Special Limits

The inequalities and limits given below are useful in establishing the derivatives of trigonometric functions.

**Theorem 1:**
If $\theta$ is measured in radians, then

$-|\theta| < \sin\theta < |\theta|$ and $-|\theta| < 1-\cos\theta < |\theta|$

**Proof**
Consider $\theta$ as an angle in standard position



The circle in the figure is a unit circle, so $|\theta|$ equals the length of the circular arc $AP$. The length of line segment $AP$ is therefore less than $|\theta|$.

Triangle $APQ$ is a right angled triangle with sides of length

$QP = |\sin\theta|$ , $AQ = |1-\cos\theta|$

From the Pythagorean theorem and the fact that $AP < |\theta|$, we get,

$$\sin^2\theta + (1-\cos\theta)^2 = AP^2 < \theta^2$$

The terms on the left side are both positive, so each is smaller than their sum and hence is less than $\theta^2$:

$$\sin^2\theta < \theta^2 \quad , \quad (1-\cos\theta)^2 < \theta^2$$

We take square roots.

$$|\sin\theta| < |\theta| \quad , \quad |1-\cos\theta| < |\theta|$$

This is equivalent to saying

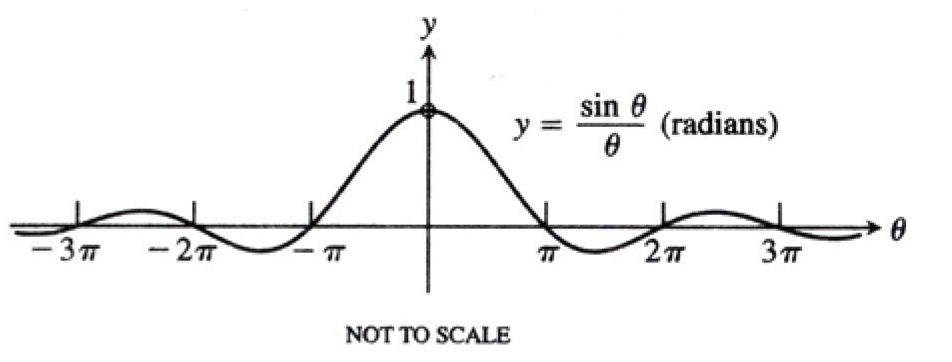
$-|\theta| < \sin\theta < |\theta|$ and $-|\theta| < 1-\cos\theta < |\theta|$

Thus the result is proved.

**Example 1:** Show that $\sin\theta$ and $\cos\theta$ are continuous at $\theta = 0$.

**Solution**
We need to show that $\lim_{\theta\to 0}\sin\theta = 0$ and $\lim_{\theta\to 0}\cos\theta = 1$. As $\theta\to 0$, both $|\theta|$ and $-|\theta|$ approach zero. The result follows immediately from the theorem 1 and the Sandwich Theorem.

The function $f(\theta) = (\sin\theta)/\theta$ graphed below appears to have a removable discontinuity at $\theta = 0$. As the figure suggests, $\lim\limits_{\theta\to 0} f(\theta) = 1$.

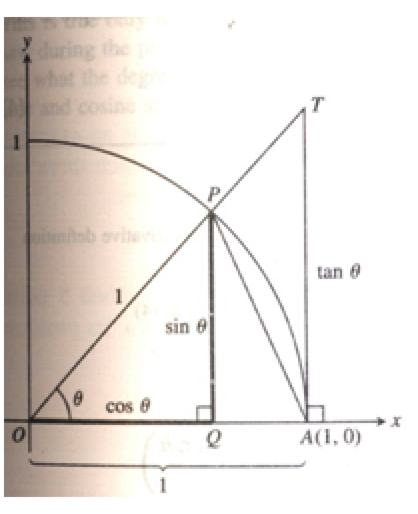



**Theorem 2:**

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1 \quad (\theta \text{ in radians})$$

**Proof:**

We prove this by showing that the right-hand and left-hand limits are both 1. To show that the right-hand limit is 1, we begin with values of $\theta$ that are positive and less than $\pi/2$.



We notice that

Area of triangle $OAP$ < area of sector $OAP$ < area of triangle $OAT$

$$\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta$$

We divide by the positive number $(1/2)\sin\theta$.

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$$

We take reciprocals, which reverses the inequalities:

$$1 > \frac{\sin\theta}{\theta} > \cos\theta$$

Since $\lim\limits_{\theta\to 0^+}\cos\theta = 1$, the Sandwich Theorem gives

$$\lim_{\theta\to 0^+}\frac{\sin\theta}{\theta} = 1$$

We observe that $\sin\theta$ and $\theta$ are both odd functions. Therefore, $f(\theta) = (\sin\theta)/\theta$ is an even function, with a graph symmetric about the $y$-axis. This symmetry implies that the left-hand limit at 0 exists and has the same value as the right-hand limit:

$$\lim_{\theta\to 0^-}\frac{\sin\theta}{\theta} = 1 = \lim_{\theta\to 0^+}\frac{\sin\theta}{\theta}$$

Therefore, $\lim\limits_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$

Thus the result is proved.

Theorem 2 can be combined with limit rules and known trigonometric identities to yield other trigonometric limits.

**Example 2: Show that $\lim\limits_{h\to 0}\frac{\cos h - 1}{h} = 0$**

**Solution:**
Using the half-angle formula $\cos h = 1 - 2\sin^2(h/2)$, we calculate

$$\lim_{h\to 0}\frac{\cos h-1}{h} = \lim_{h\to 0} -\frac{2\sin^2(h/2)}{h} = -\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\sin\theta,$$

where $\theta = h/2$

$$= -\lim_{\theta\to 0}\frac{\sin\theta}{\theta}.\lim_{\theta\to 0}\sin\theta = -(1)(0) = 0$$

**Example 3:** Show that $\lim\limits_{x\to 0}\frac{\sin 3x}{2x} = \frac{3}{2}$

**Solution:**

$$\lim_{x\to 0}\frac{\sin 3x}{2x} = \lim_{x\to 0}\left(\frac{3}{2}\right)\frac{\sin 3x}{3x}$$
$$= \frac{3}{2}.\lim_{\theta\to 0}\frac{\sin\theta}{\theta}, \quad \text{where } \theta = 3x$$
$$= \frac{3}{2}$$

## 7.9

## Derivatives of Trigonometric Functions

### Learning objectives:

- To derive the derivatives of sine and cosine functions.
- To study an example of simple harmonic motion.
- To define jerk in the motion of a body.

  And
- To practice related problems.

### Derivatives of Trigonometric Functions

**The Derivative of $y = \sin x$ :**

We have , $y = \sin x$

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} = \lim_{h\to 0}\frac{\sin x(\cos h-1)+\cos x\sin h}{h}$$

$$= \sin x\,.\lim_{h\to 0}\frac{\cos h-1}{h} + \cos x\,.\lim_{h\to 0}\frac{\sin h}{h}$$

$$= \cos x \qquad \left(\because \lim_{h\to 0}\frac{\cos h-1}{h} = 0 \text{ and } \lim_{h\to 0}\frac{\sin h}{h} = 1\right)$$

The derivative of the sine is the cosine.

$$\frac{d}{dx}(\sin x) = \cos x$$

**Example 1:**

a) $y = x^2 - \sin x \qquad \frac{dy}{dx} = 2x - \frac{d}{dx}(\sin x) = 2x - \cos x$

b) $y = x^2\sin x$

$$\frac{dy}{dx} = x^2\frac{d}{dx}(\sin x) + \sin x\frac{d}{dx}(x^2) = x^2\cos x + 2x\sin x$$

c) $y = \frac{\sin x}{x}$

$$\frac{dy}{dx} = \frac{x.\frac{d}{dx}(\sin x) - \sin x.1}{x^2} = \frac{x\cos x - \sin x}{x^2}$$

**The Derivative of $y = \cos x$ :**

We have, $y = \cos x$

$$\frac{d}{dx}(\cos x) = \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}$$

$$= \lim_{h\to 0}\frac{(\cos x\cos h - \sin x\sin h)-\cos x}{h}$$

$$= \lim_{h\to 0}\frac{\cos x(\cos h-1)-\sin x\sin h}{h}$$

$$= \lim_{h\to 0}\cos x\,.\frac{(\cos h-1)}{h} - \lim_{h\to 0}\sin x\,.\frac{\sin h}{h}$$

$$= \cos x\,.\lim_{h\to 0}\frac{(\cos h-1)}{h} - \sin x\,.\lim_{h\to 0}\frac{\sin h}{h}$$

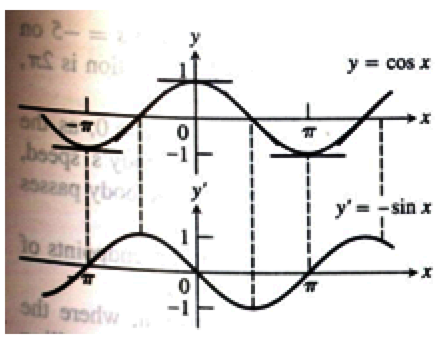
$$= \cos x\,.0 - \sin x\,.1$$

$$\left(\because \lim_{h\to 0}\frac{\cos h-1}{h} = 0\,, \lim_{h\to 0}\frac{\sin h}{h} = 1\right)$$

$$= -\sin x$$

The derivative of the cosine is the negative of the sine.

$$\frac{d}{dx}(\cos x) = -\sin x$$




The curve $y' = -\sin x$ is the graph of the tangents to the curve $y = \cos x$.

**Example 2:**

a) $y = 5x + \cos x$

$$\frac{dy}{dx} = \frac{d}{dx}(5x) + \frac{d}{dx}(\cos x) = 5 - \sin x$$

b) $y = \sin x\cos x$

$$\frac{dy}{dx} = \sin x\frac{d}{dx}(\cos x) + \cos x\frac{d}{dx}(\sin x)$$

$$= \sin x\,(-\sin x) + \cos x\,(\cos x)$$

$$= \cos^2 x - \sin^2 x$$

c) $y = \frac{\cos x}{1-\sin x}$

$$\frac{dy}{dx} = \frac{(1-\sin x)\frac{d}{dx}(\cos x) - \cos x\frac{d}{dx}(1-\sin x)}{(1-\sin x)^2}$$

$$= \frac{(1-\sin x)(-\sin x) - \cos x(0-\cos x)}{(1-\sin x)^2}$$

$$= \frac{-\sin x + \sin^2 x + \cos^2 x}{(1-\sin x)^2}$$

$$= \frac{1-\sin x}{(1-\sin x)^2}$$

$$= \frac{1}{1-\sin x}$$

## 7.10

## Continuity of Trigonometric Functions

### Learning objectives:

- To obtain the derivatives of the trigonometric functions.
- To discuss the continuity of the trigonometric functions. And
- To practice the related problems.

**The Derivatives of the Other Basic Functions**

Because $\sin x$ and $\cos x$ are differentiable functions of $x$, the related functions

$$\tan x = \frac{\sin x}{\cos x} \quad , \quad \sec x = \frac{1}{\cos x}$$

$$\cot x = \frac{\cos x}{\sin x} \quad , \quad \csc x = \frac{1}{\sin x}$$

are differentiable at every value of $x$ at which they are defined. Their derivatives, calculated from the Quotient Rule, are given by the following formulas.

$$\frac{d}{dx}(\tan x) = sec^2 x \quad , \quad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\cot x) = -csc^2 x \, , \quad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

There is minus sign in the derivative formulas for the co functions. We derive the formula for $\tan x$ below.

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{cos^2 x}$$

$$= \frac{\cos x \cos x - \sin x(-\sin x)}{cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

Similarly, other formulas can be derived.

**Example:**

Find $y''$ if $y = \sec x$.

**Solution:**

Given that $y = \sec x$

$$y' = \sec x \tan x$$

$$y'' = \sec x \frac{d}{dx}(\tan x) + \tan x \frac{d}{dx}(\sec x)$$

$$= \sec x\,(sec^2 x) + \tan x\,(\sec x \tan x)$$

$$= \sec^3 x + \sec x \tan^2 x$$

**Example:**

a. $\frac{d}{dx}(3x + \cot x) = 3 + \frac{d}{dx}(\cot x) = 3 - \csc^2 x$

b. $\frac{d}{dx}\left(\frac{2}{\sin x}\right) = \frac{d}{dx}(2 \csc x) = 2\frac{d}{dx}(\csc x) = -2 \csc x \cot x$