5.1

Rates of Change

Learning objectives:

- To define the average rate of change of a function over an interval and to understand the concept of instantaneous rate of change at a point.

  AND

- To practice the relate problems

Average speed:

First, we consider a familiar concept.

The ***average speed*** of a moving body over a time interval is the distance covered during the time interval divided by the length of the interval.

Example 1:

A rock falls from the top of a 50 m cliff.

Physical experiments show that a solid object dropped from the rest to fall freely near the surface of the earth will fall

$$y = 5t^2 \ m$$

during the first t sec.

i) Find the average speed:
   a) During the first 2 sec of fall.
   b) During the 1 sec interval between second 1 and second 2.

ii) Find the speed of the rock at $t = 1$ and $t = 2$ sec.

Solution:

The average speed of the rock during a given time interval is the change in distance Δy, divided by the length of the time interval Δt.

i) The average speed

a) For the first 2 sec: $\frac{\Delta y}{\Delta t} = \frac{5\times 2^2 - 5\times 0^2}{2-0} = 10 m/sec$

b) From second 1 to second 2:

$$\frac{\Delta y}{\Delta t} = \frac{5\times 2^2 - 5\times 1^2}{2-1} = 15 m/sec$$

ii) We can calculate the average speed of the rock over a time interval $[t_0, t_0 + h]$ having length $\Delta t = h$ as

$$\frac{\Delta y}{\Delta t} = \frac{5(t_0+h)^2 - 5t_0^2}{h}$$

We cannot use this formula to calculate the "*instantaneous*" speed at t_0 by substituting $h = 0$, because we cannot divide by 0. But we can use it to calculate average speeds over increasingly short time intervals starting at $t_0 = 1$ and $t_0 = 2$.

Length of time interval h	Average speed over interval of length h starting at $t_0 = 1$	Average speed over interval of length h starting at $t_0 = 2$
0.1	10.5	20.5
0.01	10.05	20.05
0.001	10.005	20.005
0.0001	10.0005	20.0005
.	.	.
.	.	.
.	.	.
0	10	20

The average speed on intervals starting at $t_0 = 1$ seems to approach a limiting value of 10 as the length of the interval decreases. This suggests that the rock is falling at a speed of 10 m/sec at $t_0 = 1$ sec. Similarly, the rock's speed at $t_0 = 2$ sec would appear to be 20 m/sec.

5.2

Concept of Limit

Learning objectives:

- To understand the concept of the limit of a function through examples and to give an informal definition of the limit of a function

  AND
- To practice related problems.

Concept of Limit

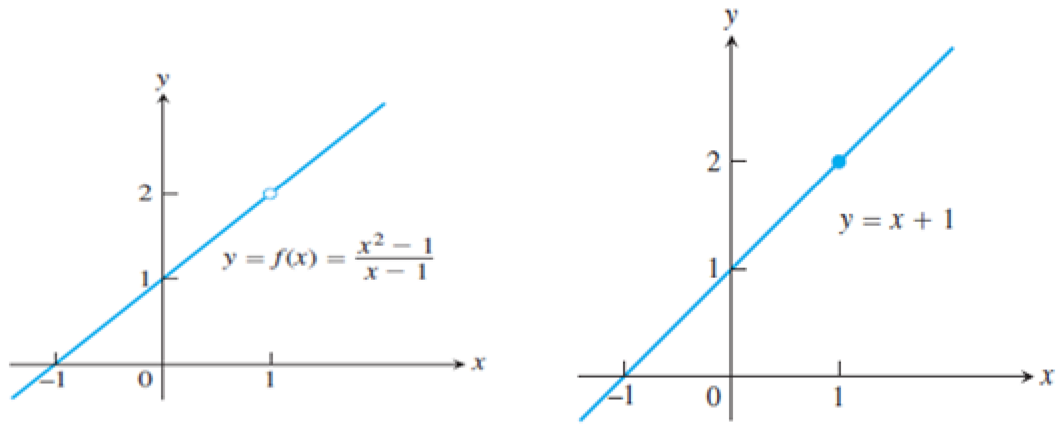
The concept of limit of a function is one of the fundamental ideas that distinguish calculus from algebra and trigonometry. First, we develop the limit intuitively and then formally.
First we look at an example:

How does the function $f(x)=\dfrac{x^2-1}{x-1}$ behave near $x=1$?

The given function f is defined for all real numbers x except $x = 1$ (since we cannot divide by zero). For any $x \neq 1$ we can simplify the function by factoring the numerator and cancelling the common factors:

$$f(x)=\frac{(x-1)(x+1)}{x-1}=x+1 \quad for \ x\neq 1$$

The graph of f is thus the line $y = x + 1$ with one point $(1,2)$ removed. This removed point is shown as a "hole" in the figure.



The graph of f is identical with the line $y = x + 1$ except at $x = 1$where f is not defined.

Even though $f(1)$ is not defined, it is clear that we can make the value of $f(x)$ *as close as we want* to 2 by choosing x *close enough* to 1.(see the following table)

Values of x Below and above 1	$f(x)=\dfrac{x^2-1}{x-1}=x+1,\ x\neq 1$
0.9	1.9
1.1	2.1
0.99	1.99
1.01	2.01
0.999	1.999
1.001	2.001
0.999999	1.999999
1.000001	2.000001

We notice that the closer x gets to1, the closer $f(x)$ seems to get 2.

We say that $f(x)$ approaches arbitrarily close to 2 as x approaches 1, or, more simply, *f(x) approaches the* ***limit* 2** *as* ***x*** *approaches* **1**. We write this as

$$\lim_{x\to 1} f(x)=2 \quad or \quad \lim_{x\to 1}\frac{x^2-1}{x-1}=2$$

5.3

Rules for Finding Limits

Learning objectives:

- To state the properties of limits and to apply them to polynomial and rational functions
- To state the Sandwich Theorem

 And

- To find the limits of functions by different techniques

Properties of Limits

Here we state, rules to calculate the limits of functions that are the arithmetic combination of functions whose limits are known.

If L, M, *c* and *k* are real numbers and $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$ then,

1. Sum Rule: $\lim_{x\to c}[f(x)+g(x)] = L+M$

i. e., The limit of the sum of two functions is the sum of their limits

2. Difference Rule: $\lim_{x\to c}[f(x)-g(x)] = L-M$

i.e., The limit of the difference of two functions is the difference of their limits.

3. Product Rule: $\lim_{x\to c}[f(x)\cdot g(x)] = L\cdot M$

i.e., The limit of the product of two functions is the product of their limits.

4. Constant Multiple Rule:

$$\lim_{x\to c} kf(x) = kL \qquad (\text{any number } k)$$

i.e., The limit of a constant times a function is that constant times the limit of the function.

5. Quotient Rule: $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{M}, \qquad M \neq 0$

i.e., The limit of the quotient of two functions is the quotient of their limits, provided the limit of the denominator is not zero.

6. Power Rule: If *m* and *n* are integers, then

$\lim_{x\to c}[f(x)]^{\frac{m}{n}} = L^{\frac{m}{n}}$, provided $L^{\frac{m}{n}}$ **is a real number.**

i.e., The limit of any rational power of a function is that power of the limit of the function

Example 1: Find $\lim_{x\to c} \frac{x^3+4x^2-3}{x^2+5}$

Solution:

$$\lim_{x\to c} x^3+4x^2-3 = \lim_{x\to c} x^3 + \lim_{x\to c} 4x^2 - \lim_{x\to c} 3$$
(Sum and difference rule)

$$= \lim_{x\to c} x^3 + 4\lim_{x\to c} x^2 - 3$$
(Constant multiple rule)

$$= \left(\lim_{x\to c} x\right)^3 + 4\left(\lim_{x\to c} x\right)^2 - 3$$
(Power rule or product rule)

$$= c^3+4c^2-3 \quad \left(\because \lim_{x\to c} x = c, \lim_{x\to c} k = k\right)$$

$$\lim_{x\to c} x^2+5 = \lim_{x\to c} x^2 + \lim_{x\to c} 5$$ (Sum rule)

$$= \left(\lim_{x\to c} x\right)^2 + \lim_{x\to c} 5$$ (Power rule)

$$= c^2+5 \quad \left(\because \lim_{x\to c} x = c, \lim_{x\to c} k = k\right)$$

Now $$\lim_{x\to c} \frac{x^3+4x^2-3}{x^2+5} = \frac{\lim_{x\to c} x^3+4x^2-3}{\lim_{x\to c} x^2+5}$$ (quotient rule)

$$= \frac{c^3+4c^2-3}{c^2+5} \quad \left(\because c^2+5 \neq 0\right)$$

Example 2: Find $\lim_{x\to -2} \sqrt{4x^2-3}$

Solution:

$$\lim_{x\to -2} \sqrt{4x^2-3} = \sqrt{\lim_{x\to -2}\left(4x^2-3\right)}$$ (Power rule)

$$= \sqrt{\lim_{x\to -2} 4x^2 - \lim_{x\to -2} 3}$$ (Difference rule)

$$= \sqrt{4\lim_{x\to -2} x^2 - 3}$$ (Constant multiple rule)

$$= \sqrt{4\left(\lim_{x\to -2} x\right)^2 - 3}$$ (Power rule)

$$= \sqrt{4(-2)^2-3} = \sqrt{13}$$

5.4

Formal Definition of Limit

Learning objectives

- To give a formal definition of the limit of a function and to prove some properties of limits.
  And
- To solve related problems.

Formal Definition of Limit

We look at an example of determining the input values x that ensure the output y near a ***target*** value.

Example 1:

How close to $x_0 = 4$ must we hold the input x to be sure that the output $y = 2x - 1$ lies within 2 units of $y_0 = 7$?

Solution:

The problem simply means:

For what values of x is $|y - 7| < 2$.

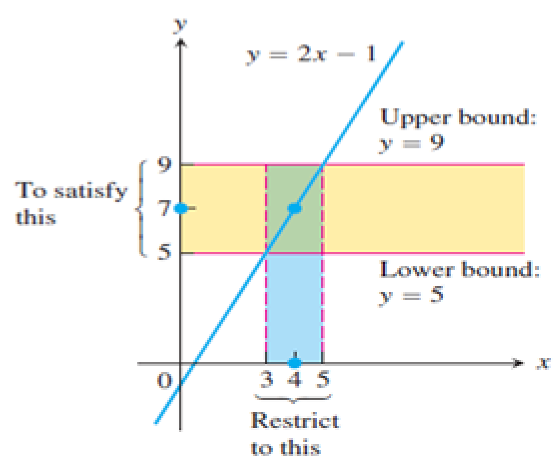
We have $|y-7|=|2x-1-7|=|2x-8|$. The question becomes:

What values of x satisfy the inequality $|2x - 8| < 2$?

We solve the inequality:

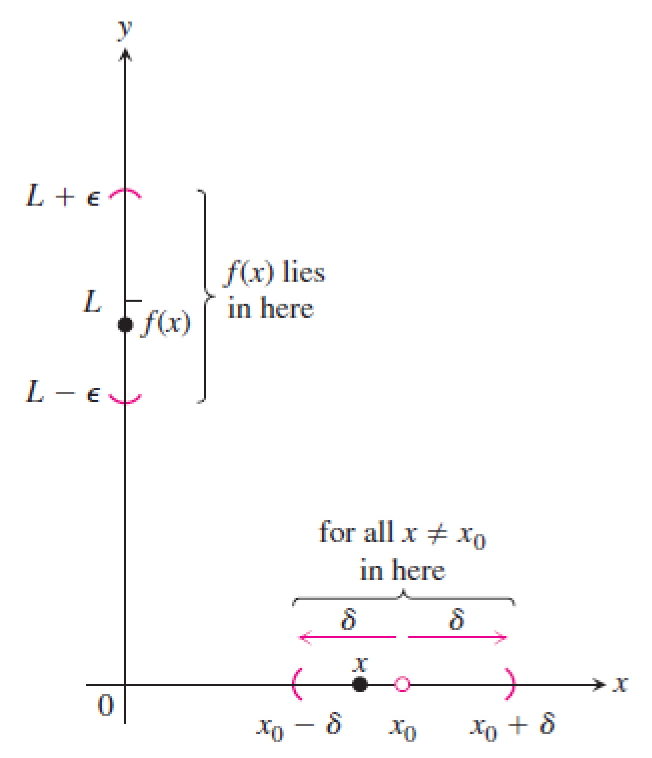
$$|2x-8|<2$$
$$\Rightarrow -2<2x-8<2 \Rightarrow 6<2x<10$$
$$\Rightarrow \quad 3<x<5 \quad \Rightarrow -1<x-4<1$$

Keeping x within 1 unit of $x_0 = 4$ will keep y within 2 units of $y_0 = 7$.



In a target-value problem, we determine how close to hold a variable x to a particular value x_0 to ensure that the outputs $f(x)$ lie within a prescribed interval about a target value L.

To show that the limit of $f(x)$ as $x \to x_0$ actually equals L, we must be able to show that the gap between $f(x)$ and L can be made less than any prescribed error, no matter how small, by holding x close enough to x_0.



For every ε, if we can find a δ, then we say $f(x)$ has limit L as x tends to x_0.

Definition

Let $f(x)$ be defined on an open interval about x_0, except possibly at x_0 itself. We say that $f(x)$ approaches the limit L as x approaches x_0, and write $\lim_{x\to x_0} f(x)=L$, if, for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x

$$0<|x-x_0|<\delta \Rightarrow |f(x)-L|<\varepsilon$$

Note: If $\lim_{x\to x_0} f(x)=L$ exists then it is unique.

The now accepted ε,δ definition of limit was formulated by German mathematician Weierstrass in the middle of the nineteenth century.

The formal definition of limit does not tell how to find the limit of a function, but it enables us to verify that a suspected limit is correct. However, the real purpose of the definition is not to do calculations like this, but rather to prove general theorems so that the calculation of specific limits can be simplified.

Example 2: Show that $\lim_{x\to 1}(5x-3)=2$.

Solution:

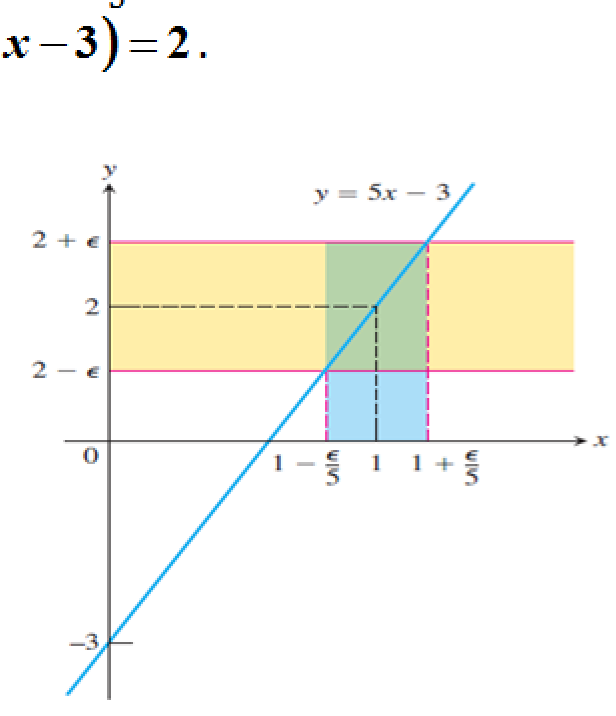
For a given $\varepsilon > 0$, we have to find a suitable $\delta > 0$ so that if

$$0<|x-1|<\delta \text{ then } |f(x)-2|<\varepsilon.$$

We find δ by working backwards from the ε inequality:

$$|f(x)-2|=|(5x-3)-2|=|5x-5|<\varepsilon$$
$$\text{i.e., } 5|x-1|<\varepsilon$$
$$\text{i.e., } |x-1|<\varepsilon/5$$

Thus, we can take $\delta = \frac{\varepsilon}{5}$ or any smaller positive value. This proves that $\lim_{x\to 1}(5x-3)=2$.



Note:

The process of finding a $\delta>0$ such that for all x

$$0<|x-x_0|<\delta \quad \Rightarrow \quad |f(x)-L|<\varepsilon$$

can be accomplished in two steps.

1) We solve the inequality $|f(x)-L|<\varepsilon$ to find an open interval (a,b) about x_0 on which the inequality holds for all $x \neq x_0$.

2) We find a value of $\delta>0$ that places the open interval $(x_0-\delta, x_0+\delta)$ centered at x_0 inside the interval (a,b). The inequality $|f(x)-L|<\varepsilon$ will hold for all $x \neq x_0$ in this δ-interval.

5.5.

Extension of the Limit concept

Learning objectives:

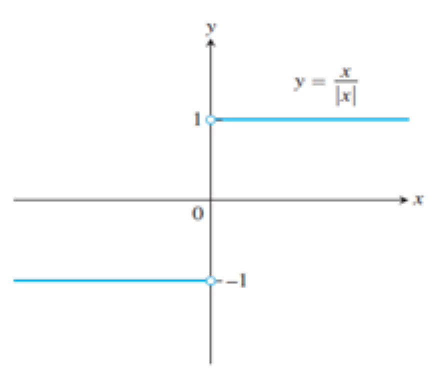
- To define right and left hand limits
- To define the limit in terms of one sided limits
  And
- To practice related problems.

Now we extend the concept of limit to ***one-sided limits***, which are limits as x approaches a from the left-hand side(where $x < a$) or the right-hand side (where $x > a$) only.

One Sided Limits

To have a limit L as x approaches a, a function f must be defined on both sides of a, and its value $f(x)$ must approach L as x approaches a from either side. Because of this, ordinary limits are sometimes called **two-sided** limits.

It is possible for a function to approach a limiting value as x approaches a from only one side, either from the right or from the left. In this case we say that f has a **one-sided** limit at a. The function $f(x)=\frac{x}{|x|}$ graphed below has limit 1 as x approaches zero from the right, and limit -1 as x approaches zero from the left.



Definition

Let $f(x)$ defined on an interval (a, b) where $a < b$. If $f(x)$ approaches arbitrarily close to L as x approaches a from within that interval, then we say that f has **right-hand limit** L at a, and we write

$$\lim_{x \to a^+} f(x) = L$$

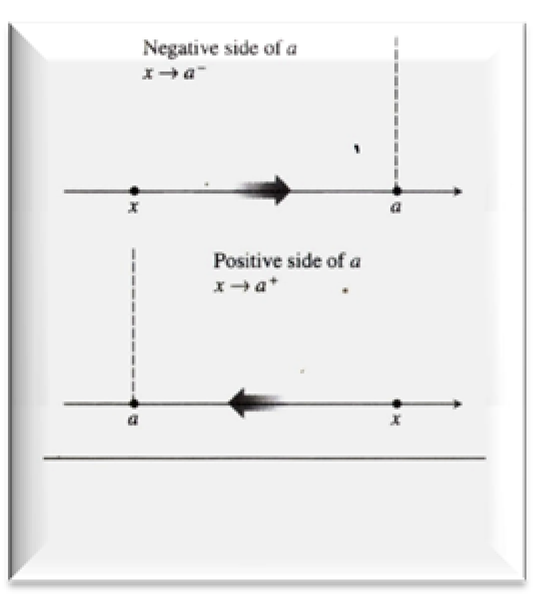
Let $f(x)$ be defined on an interval (c, a) where $c < a$. If $f(x)$ approaches arbitrarily close to M as x approaches a from within that interval, then we say that f has **left-hand limit** M at a, and we write

$$\lim_{x \to a^-} f(x) = M$$

For the function $f(x)=\frac{x}{|x|}$ in the figure above, we have

$$\lim_{x \to 0^+} f(x) = 1 \quad \text{and} \quad \lim_{x \to 0^-} f(x) = -1$$

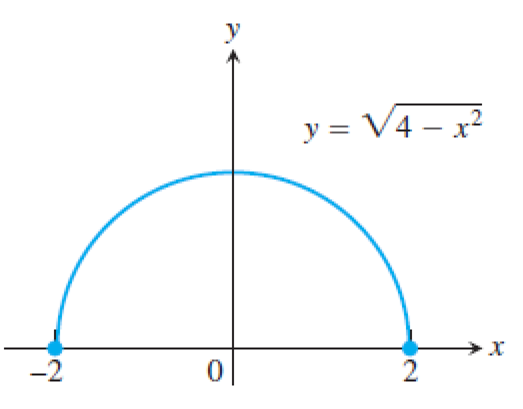
A function cannot have an ordinary limit at an endpoint of its domain, but it can have a one-sided limit.



The symbol $x \to a^-$ means x approaches a from the negative side of a, through values less than a.

The symbol $x \to a^+$ means x approaches a from the positive side of a, through values greater than a.

Example 1: The domain of $f(x)=\sqrt{4-x^2}$ *is* $[-2,2]$; its graph is semi-circle shown below.



$$\lim_{x \to -2^+} f(x) = 0 \ , \lim_{x \to 2^-} f(x) = 0$$

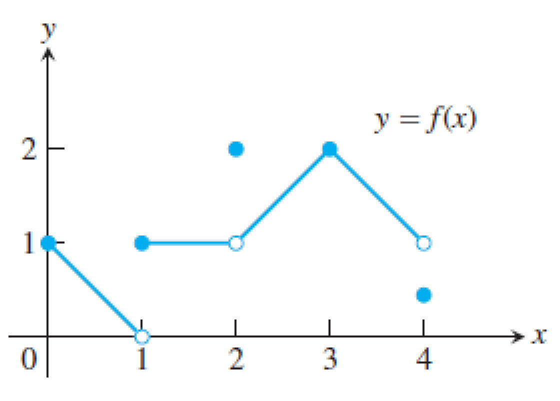
The function does not have a left-hand limit at $x = -2$ or a right-hand limit at $x = 2$. It does not have ordinary two-sided limits at either -2 or 2.

One-sided versus two-sided Limits

A function $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there, and these one-sided limits are equal:

$$\lim_{x \to c} f(x) = L \iff \lim_{x \to c^-} f(x) = L \ \ and \ \ \lim_{x \to c^+} f(x) = L$$

Example 2: All of the following statements about the function graphed in the figure below are true.



At $x = 0$

$\lim_{x \to 0^+} f(x) = 1$

$\lim_{x \to 0^-} f(x)$ and $\lim_{x \to 0} f(x)$ do not exist. The function is not defined to the left of $x = 0$.

At $x = 1$

$\lim_{x \to 1^+} f(x) = 0$ even though $f(1) = 1$

$\lim_{x \to 1^+} f(x) = 1$

$\lim_{x \to 1} f(x)$ does not exist. The right-hand and left-hand limits are not equal.

At $x = 2$

$\lim_{x \to 2^-} f(x) = 1$

$\lim_{x \to 2^+} f(x) = 1$

$\lim_{x \to 2} f(x) = 1$ even though $f(2) = 2$.

At $x = 3$

$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = \lim_{x \to 3} f(x) = f(3) = 2$

At $x = 4$

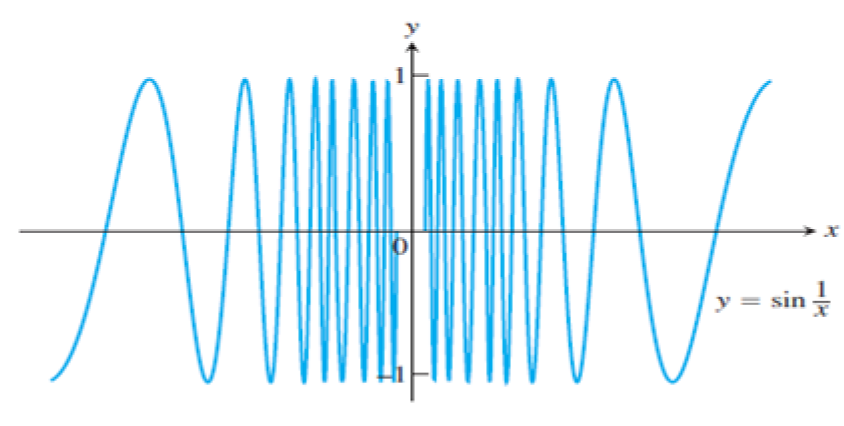
$\lim_{x \to 4^-} f(x) = 1$ even though $f(4) \neq 1$.

$\lim_{x \to 4^+} f(x)$ and $\lim_{x \to 4} f(x)$ do not exist. The function is not defined to the right of $x = 4$.

At every other point a in $[0,4]$, $f(x)$ has limit $f(a)$.

The function $y = \sin\left(\frac{1}{x}\right)$ has neither a right-hand nor a left-hand limit as x approaches zero. This can be seen from the following observations.

As x approaches zero, its reciprocal $\frac{1}{x}$ grows without bound and the value of $\sin(1/x)$ cycle repeatedly from -1 to 1. There is no single number L that the function's values stay increasingly close to as x approaches zero. This is true even if we restrict x to positive or negative values. The function has neither a right-hand limit nor a left-hand limit at $x = 0$.



The formal definition of two-sided limits can be easily modified for one-sided limits.

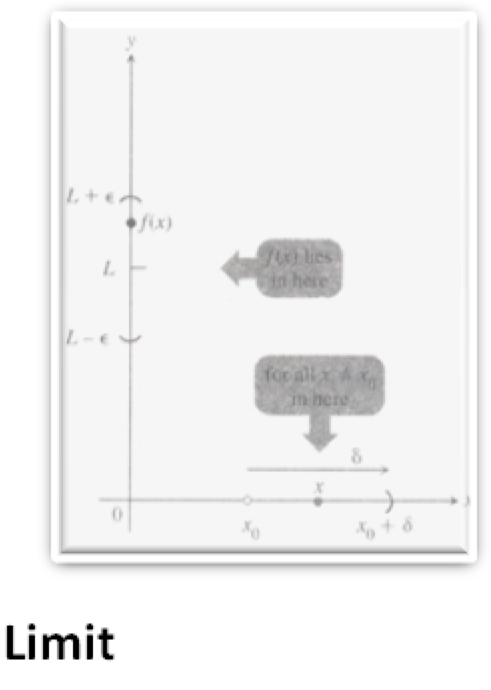
Right-hand Limit

We say that $f(x)$ has right-hand limit L at x_0, and write

$$\lim_{x \to x_0^+} f(x) = L$$

if for every number $\varepsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

$$x_0 < x < x_0 + \delta \quad \Rightarrow \quad |f(x) - L| < \varepsilon$$



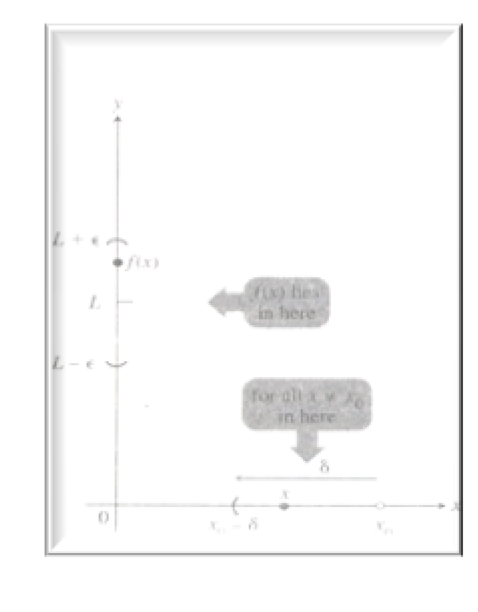
Left-hand Limit

We say that $f(x)$ has left-hand limit L at x_0, and write

$$\lim_{x \to x_0^-} f(x) = L$$

if for every number $\varepsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

$$x_0 - \delta < x < x_0 \quad \Rightarrow \quad |f(x) - L| < \varepsilon$$



5.6

Infinite Limits

Learning objectives:

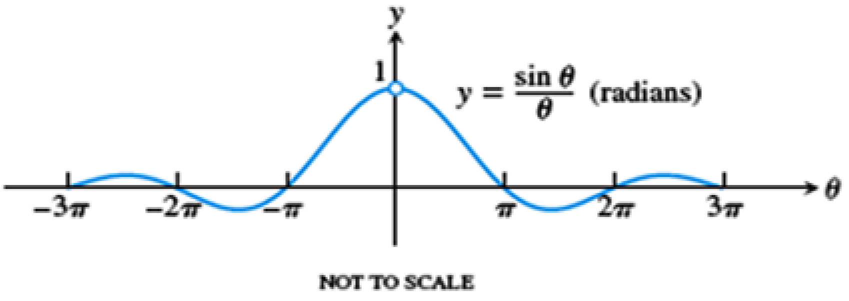
- To study $\lim_{\theta \to 0} \frac{sin\theta}{\theta}$.
- To define limits of a function at $\pm\infty$
- To define infinite limits of functions

AND

- To practice related problems.

Limits involving $\frac{\sin\theta}{\theta}$

We have already noted that $\lim_{\theta\to 0} sin\,\theta = 0$ and $\lim_{\theta\to 0} cos\,\theta = 1$. We now, take up $\lim_{\theta\to 0}\frac{\sin\theta}{\theta}$, where θ is measured in Radian measure. It may be seen $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$ from the following figure



The graph of $f(\theta) = \frac{(\sin\theta)}{\theta}$

Notice that $\sin\theta$ and θ are odd functions. Therefore

$f(\theta) = \frac{\sin\theta}{\theta}$ is an even function with a graph symmetry about the y −axis (see the above fig). This symmetry implies that the left-hand limit at 0 exists and has the same value as the right hand limit:

$$\lim_{\theta\to 0^-}\frac{\sin\theta}{\theta} = 1 = \lim_{\theta\to 0^+}\frac{\sin\theta}{\theta}$$

Therefore, $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$

We prove the above result algebraically in a subsequent module.

Example 1:

Find (i) $\lim_{x\to 0}\frac{cosx-1}{x}$ (ii) $\lim_{x\to 0}\frac{\sin 3x}{2x}$

Solution:

$$(i)\ \lim_{x\to 0}\frac{cosx-1}{x} = \lim_{x\to 0}\frac{-2\,sin^2\frac{x}{2}}{x} = -\lim_{x\to 0}\frac{sin\frac{x}{2}}{\left(\frac{x}{2}\right)}.\lim_{x\to 0} sin\frac{x}{2}$$

$$= -1.0 = 0$$

$$(ii)\ \lim_{x\to 0}\frac{\sin 3x}{2x} = \lim_{x\to 0}\left(\frac{3}{2}\right)\frac{\sin 3x}{3x}$$

$$= \frac{3}{2}.\lim_{x\to 0}\frac{sin\,3x}{3x}$$

Put $\theta = 3x$. Now, $\theta \to 0$ as $x \to 0$.

$$= \left(\frac{3}{2}\right).\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = \frac{3}{2}.1 = \frac{3}{2}$$