# 2.1
## Vector Addition and Subtraction

The mathematical concept of vector turns out to be very useful for the description of position, velocity, and acceleration in two- or three-dimensional motion.

### Physical Quantities

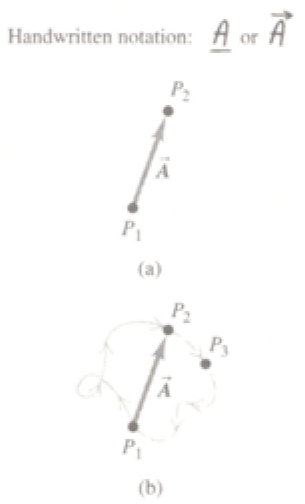
A *physical quantity* is a number that is used to describe a physical phenomenon quantitatively. Some physical quantities, such as time, temperature, mass, density, and electric charge, can be described simply by a single number with a unit. For example, the temperature at a point of a body is 60° C. But many other important quantities have a *direction* associated with them and cannot be described by a single number.

To describe the motion of an airplane, for example, we must say not only how fast the plane is moving, but also in what direction. The speed of airplane combined with its direction of motion together constitutes a quantity called *velocity*. Another example is *force*, which means a push or pull exerted on a body. We need to specify not only the amount of push or pull but also the direction along which the push or pull is applied.

### Scalar and Vector Quantities

When a physical quantity is described by a single number, it is called a **scalar quantity**. In contrast, a **vector quantity** has both a magnitude and a direction in space. Calculations with scalar quantities use the operations of ordinary arithmetic. For example, 6 kg + 3 kg = 9 kg. However, combining vectors requires a different set of operations.

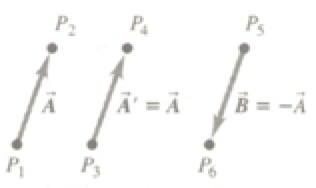
We consider the vector quantity **displacement**. Displacement is simply a change in position of a particle. In the figure below, we represent the change of position from point $P_1$ to $P_2$ by a line from $P_1$ to $P_2$ with an arrowhead at $P_2$ to represent the direction of the motion. Displacement is a vector quantity because we must state not only how far the particle moves, but also the direction along which it moves. The directed line segment $\vec{P_1P_2}$ is the **displacement vector** of the particle. A dispacement vector has a *length* and a *direction*




We usually represent a vector quantity such as displacement by a single letter, such as $\vec{A}$. In print, it is conventional to use *vector symbols in boldface without an arrow above them*, for example, as **A**. In handwriting, *vector symbols are written in light italic with an arrow above them*, for example as $\vec{A}$. When drawing any vector, we always draw line with an arrowhead at its tip. The length of the line shows the vector's magnitude, and the direction of the line shows the vector's direction.

Displacement is always a straight-line segment, directed from the starting point to the end point, even though the actual path of the particle may be curved. In figure (b) the particle moves along the curved path shown from $P_1$ to $P_2$, but the displacement is still the vector **A**. The displacement is not related directly to the total *distance* traveled. If the particle were to continue on to $P_3$ and then return to $P_1$, the displacement for the entire trip would be *zero*.

If two vectors have the same direction, they are **parallel**. If they have the same magnitude *and* the same direction, they are *equal*, no matter where they are located in space. The vector $\mathbf{A}'$ from point $P_3$ to point $P_4$ in the figure below has the same length and direction as the vector **A** from $P_1$ to $P_2$.




These two displacements are equal, even though they start at different points. We write this as $\mathbf{A} = \mathbf{A}'$. Two vector quantities are equal only when they have the same magnitude *and* the same direction.

We define the **negative of a vector** as a vector having the same magnitude as the original vector but the opposite direction. The negative of vector quantity **A** is denoted by **–A**. The relation between $\mathbf{A}$ and $\mathbf{B}$ may be written as $\mathbf{A} = -\mathbf{B}$ or $\mathbf{B} = -\mathbf{A}$. When two vectors **A** and **B** have opposite directions, whether their magnitudes are the same or not, we say that they are **antiparallel**. The magnitude of a vector **A** is denoted by $A$ (light italic) *or* $|\mathbf{A}|$. By definition the magnitude of a vector quantity is a scalar quantity (a number) and is *always positive*.

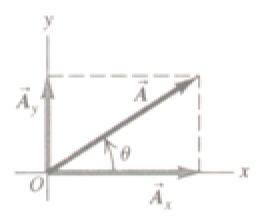
When drawing diagrams with vectors, we will use a scale in which the distance on the diagram is proportional to the magnitude of the vector. For example, a displacement of 5 km might be represented in a diagram by a vector 1 cm long. In a diagram for force vectors we might use a scale in which a vector that is 1 cm long represents a force of magnitude 5 N. A 20-N force would then be represented by a vector 4 cm long, with the appropriate direction.

## 2.2

## Vector Components

### Cartesian Coordinate System

We consider a rectangular (Cartesian) coordinate system of axes.



We draw a vector with its tail at O, the origin of the coordinate system. We can represent any vector lying in the *xy*-plane as the sum of a vector parallel to the *x*-axis and a vector parallel to the *y*-axis. These two vectors are labeled $\vec{A}_x$ and $\vec{A}_y$ in the figure above (we will drop the use of arrow with boldface letters); they are called the component vectors of vector **A**, and their vector sum is equal to **A**.

$$\mathbf{A} = \mathbf{A}_x + \mathbf{A}_y$$

In essence, we have broken the vector **A** into two perpendicular vectors that are parallel to the coordinate axes. This process is called the **decomposition** of vector **A** into its component vectors.

### Components of Vectors

By definition, each component vector lies along a coordinate-axis direction. Thus we need only a single number to describe each one. When the component vector $\mathbf{A}_x$ points in the positive *x*-direction, we define the number $A_x$ to be equal to the magnitude of $\mathbf{A}_x$. We place a negative sign in front of the magnitude $A_x$ if $\mathbf{A}_x$ points in the negative *x*-direction. We define the number in $A_y$ the same way. The two numbers $A_x$ and $A_y$ are called the **components** of the vector **A**.

We can calculate the components of the vector **A** if we know its magnitude *A* and its direction. We will describe the direction of a vector by its angle relative to some reference direction. In the figure above, the reference direction is the positive *x*-axis, and the angle between vector **A** and the positive *x*-axis is θ. We imagine that the vector **A** originally lies along the +*x*-axis and that we then rotate it to its correct direction. If this rotation is from the +*x*-axis toward the +*y*-axis, then θ is *positive*; If the rotation is from the +*x*-axis toward the -*y*-axis, θ is *negative*. Thus the +*y*-axis is at an angle of 90°, the –*x*-axis is at 180° and the –*y*-axis is at 270° (or -90°). If θ is measured this way, we have

$$\frac{A_x}{A} = \cos\theta, \quad \frac{A_y}{A} = \sin\theta$$

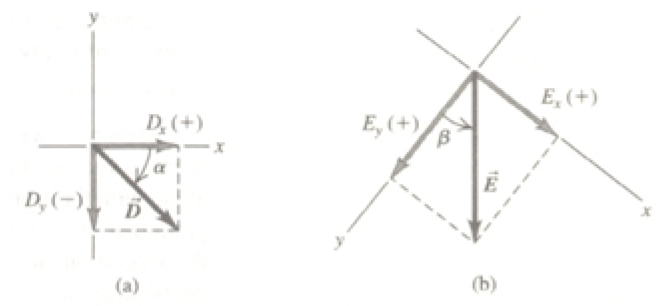
$$A_x = A\cos\theta$$

$$A_y = A\sin\theta$$

### Example

i) What are the *x*- and *y*- components of the vector **D** in fig (a) below? The magnitude of the vector is D = 3.00 m and the angle α = 45°.

ii) What are the *x*- and *y*- components of the vector **E** in fig (b) below? The magnitude of the vector is E = 4.50 m and the angle β = 37.0°.



i)

$$\theta = -\alpha = -45^\circ$$

$$D_x = D\cos\theta = 3.00\times\cos(-45^\circ) = +2.1\text{m}$$

$$D_y = D\sin\theta = 3.00\times\sin(-45^\circ) = -2.1\text{m}$$

ii)

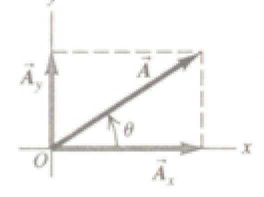
$$\theta = 90.0^\circ - \beta = 90.0^\circ - 37.0^\circ = 53.0^\circ$$

$$E_x = E\cos\theta = 4.50\times\cos(53^\circ) = +2.71\text{m}$$

$$E_y = E\sin\theta = 4.50\times\sin(53^\circ) = +3.59\text{m}$$

### Vector Addition

We can describe a vector completely by giving either its magnitude and direction or its *x*- and *y*- components. We can find the magnitude and direction if we know the components.

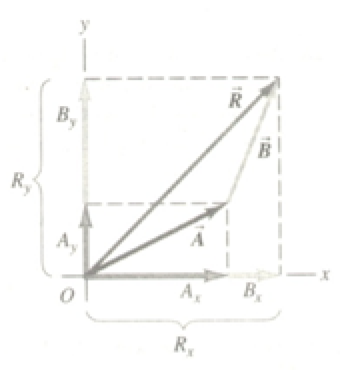


By applying the Pythagorean Theorem, we find that the magnitude of a vector **A** is given by

$$A = \sqrt{A_x^2 + A_y^2}$$

$$\tan\theta = \frac{A_y}{A_x}$$

We can use components to calculate the vector sum (resultant) of two or more vectors. The figure below shows two vectors **A** and **B** and their vector sum **R**, along with the *x*- and *y*- components of all three vectors.



As seen from the diagram, the *x*-component $R_x$ of the vector sum is simply the sum $(A_x + B_x)$ of the *x*-components of the vectors being added. The same is true for the *y*-components.

$$R_x = A_x + B_x, \quad R_y = A_y + B_y$$

From these components, we can compute the magnitude and direction of the resultant vector **R**.

This procedure for finding the sum of two vectors is applicable to any number of vectors.

Also, the component method works for vectors having any direction in space. We introduce a *z*-axis perpendicular to the *xy*-plane; then in general a vector **A** has components $A_x, A_y, A_z$ in the three coordinate directions. The magnitude *A* is given by

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

## 2.3

## Unit Vectors

Vectors can be expressed in terms of **unit vectors** along the coordinate axes.

A unit vector is a vector that has a magnitude of 1, with no units. Its only purpose is to *point*; that is, to describe a direction in space. Unit vectors provide a convenient notation for expressions involving components of vectors. It is usual to include a caret or "hat" (^) in the symbol for a unit vector to signify that its magnitude is equal to 1, such as $\hat{\mathbf{i}}$ (meaning that $\hat{\mathbf{i}}$ is a vector of unit magnitude). However, we use this "hat" notation sparingly whenever clarity requires it.

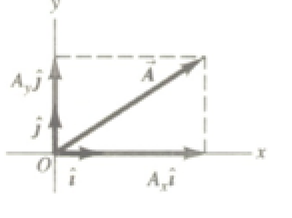
In an *xy*-coordinate system we define a unit vector $\mathbf{i}$ that points in the direction of the positive *x*-axis and a unit vector $\mathbf{j}$ that points in the direction of the positive *y*-axis. Then we can express the relationship between the component vectors and components as follows:

$$\mathbf{A}_x = A_x\mathbf{i}$$
$$\mathbf{A}_y = A_y\mathbf{j}$$

Similarly, we can write vector **A** in terms of its components as

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j}$$

The above equations are vector equations; each term, such as $A_x\mathbf{i}$, is a vector quantity.



When two vectors **A** and **B** are represented in terms of their components, we can express the vector **R** using unit vectors as follows:

$$\mathbf{A} = A_x\hat{i} + A_y\hat{j}$$
$$\mathbf{B} = B_x\hat{i} + B_y\hat{j}$$
$$\mathbf{R} = \mathbf{A} + \mathbf{B}$$
$$= (A_x\hat{i} + A_y\hat{j}) + (B_x\hat{i} + B_y\hat{j})$$
$$= (A_x + B_x)\hat{i} + (A_y + B_y)\hat{j}$$
$$= R_x\hat{i} + R_y\hat{j}$$

where

$$R_x = A_x + B_x$$
$$R_y = A_y + B_y$$

This result is the same as the result we obtained in the previous module.

Thus, the single vector equation

$$\mathbf{R} = \mathbf{A} + \mathbf{B}$$

is equivalent to the two component equations

$$R_x = A_x + B_x$$
$$R_y = A_y + B_y$$

If the vectors do not all lie in the $xy$ plane, then we need a third component.

For representing vectors lying in the space, we introduce a third unit vector $\mathbf{k}$ that points in the direction of the positive *z*-axis. We then have

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$$
$$\mathbf{B} = B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}$$
$$\mathbf{R} = \mathbf{A} + \mathbf{B}$$
$$= (A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}) + (B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k})$$
$$= (A_x + B_x)\mathbf{i} + (A_y + B_y)\mathbf{j} + (A_z + B_z)\mathbf{k}$$
$$= R_x\mathbf{i} + R_y\mathbf{j} + R_z\mathbf{k}$$

The advantage of expressing vectors in terms of unit vectors is that we can then manipulate them algebraically, as illustrated in the example below. This method of vector addition is called **algebraic addition**.

### Example

Given the two displacements

$$\mathbf{D} = (6\mathbf{i} + 3\mathbf{j} - \mathbf{k})\,\mathrm{m} \quad \text{and} \quad \mathbf{E} = (4\mathbf{i} - 5\mathbf{j} + 8\mathbf{k})\,\mathrm{m},$$

find the magnitude of the displacement 2**D** – **E**.

### Solution

Let **F** = 2**D** – **E**

$$\mathbf{F} = 2(6\mathbf{i} + 3\mathbf{j} - \mathbf{k}) - (4\mathbf{i} - 5\mathbf{j} + 8\mathbf{k})$$
$$= (12 - 4)\mathbf{i} + (6 + 5)\mathbf{j} + (-2 - 8)\mathbf{k}$$
$$= 8\mathbf{i} + 11\mathbf{j} - 10\mathbf{k}$$
$$\mathrm{F} = \sqrt{8^2 + 11^2 + (-10)^2} = 17\ \mathrm{m}$$

We can obtain a unit vector in a direction parallel to any given vector $\mathbf{A}$ by dividing the vector $\mathbf{A}$ by its magnitude $A$. Such a unit vector is often denoted with a hat above the original vector:

$$\hat{\mathbf{A}} = \frac{\mathbf{A}}{A}$$

In terms of explicit notation using unit vectors along the coordinate axes, we write

$$\hat{\mathbf{A}} = \frac{1}{A}(A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k})$$

where

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

From this form of expression, we see that $\hat{\mathbf{A}}$ has a magnitude identically equal to 1.

There is also a compact notation for a vector. The vector $\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$ may be compactly written as

$$\mathbf{A} = (A_x, A_y, A_z)$$

where the $x$, $y$, and $z$ components are placed in order between parentheses. For example, the displacement vector $\mathbf{D} = 6\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ of the previous example may be written as

$$\mathbf{D} = (6, 3, -1)$$

Howver, the unit-vector form is most often used.

**Example**: Consider the following three vectors

$$\mathbf{A} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$$
$$\mathbf{B} = \mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$$
$$\mathbf{C} = -\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$$

Find a unit vector in a direction parallel to the resultant vector $\mathbf{D} = \mathbf{A} + \mathbf{B} + \mathbf{C}$.

***Solution***: When the vectors in a sum are expressed in terms of unit vectors, we can manipulate them algebraically, by collecting terms involving the same unit vectors:

$$\mathbf{D} = \mathbf{A} + \mathbf{B} + \mathbf{C} = (2\mathbf{i} + 2\mathbf{j} - \mathbf{k}) + (\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}) + (-\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$$
$$= (2\mathbf{i} + \mathbf{i} - \mathbf{i}) + (2\mathbf{j} - 3\mathbf{j} + 2\mathbf{j}) + (-\mathbf{k} + 3\mathbf{k} + 2\mathbf{k})$$
$$= 2\mathbf{i} + \mathbf{j} + 4\mathbf{k}$$

The components of the resultant are given by

$$D_x = 2 \qquad D_y = 1 \qquad D_z = 4$$

The magnitude $D$ of the resultant is given by

$$D = \sqrt{D_x^2 + D_y^2 + D_z^2}$$
$$= \sqrt{2^2 + 1^2 + 4^2} = \sqrt{21} = 4.6$$

The unit vector along a direction parallel to the vector $\mathbf{D}$ is given by

$$\hat{\mathbf{D}} = \frac{\mathbf{D}}{D} = \frac{1}{D}(D_x\mathbf{i} + D_y\mathbf{j} + D_z\mathbf{k})$$
$$= \frac{1}{4.6}(2\mathbf{i} + \mathbf{j} + 4\mathbf{k})$$

Module 2.3

**Unit vectors**

(Home work)

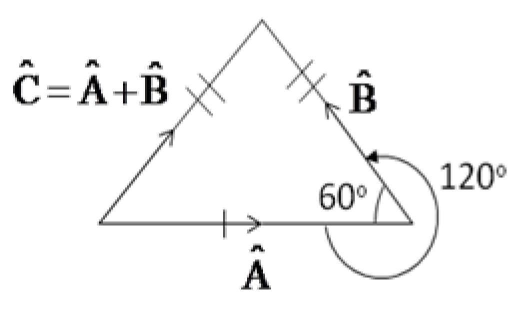
[Work out the following with proper reasoning]

1) Can there be a unit vector in any direction other than x, y, z directions? Give an example.

**Answer:** Yes, $\hat{A} = \dfrac{A_x\bar{i} + A_y\bar{j} + A_z\bar{k}}{\sqrt{A_x^2 + A_y^2 + A_z^2}}$

2) If the sum of two unit vectors is also a unit vector, what is the angle between the two unit vectors?

**Answer:** -60° or 120°




3) If $(0.5\bar{i} + 0.8\bar{j} + n\bar{k})$ is a unit vector, what is the value of n?

**Answer:** $n = \pm\sqrt{0.11}$

4) Three forces $\bar{A} = (i + \bar{j} + \bar{k})$, $\bar{B} = (2\bar{i} - \bar{j} + 3\bar{k})$ and $\bar{C}$ act on a body and keep it in equilibrium. What is the magnitude and direction of $\bar{C}$?

**Answer:** 5, $\bar{C}$ is opposite to $\bar{A} + \bar{B}$)

5) Find the angles made by the vector $2\bar{i} + 2\bar{j} + 2\bar{k}$ with x, y and z axes.

**Answer:** $\alpha = \cos^{-1}\frac{1}{\sqrt{3}}, \beta = \cos^{-1}\frac{1}{\sqrt{3}}, \gamma = \cos^{-1}\frac{1}{\sqrt{3}}$

## 2.4

## Dot Product

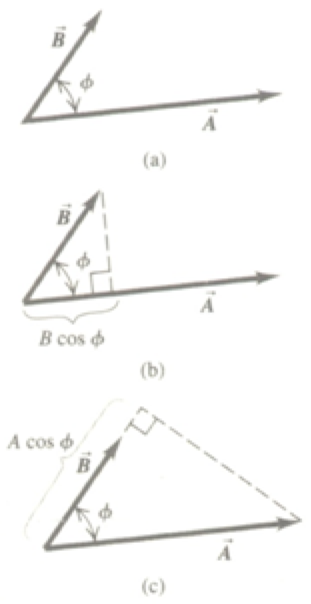
Vector addition was intimately related to the problem of combining displacements. It will be used to calculate many other vector quantities.

Products of vectors are useful for expressing many physical relationships. Vectors are not ordinary numbers, so ordinary multiplication is not directly applicable to vectors. There are two kinds of products of vectors. One is dot product, also called the scalar product, yields a result that is a scalar quantity. The second is cross product, also called the vector product, yields another vector.

In this module, we learn about the dot product, and in the next module, we study the cross product.

The **scalar product** of two vectors $\mathbf{A}$ and $\mathbf{B}$ is denoted by $\mathbf{A}\cdot\mathbf{B}$. Because of this notation, the scalar product is also called the **dot product**.

We draw two vectors $\mathbf{A}$ and $\mathbf{B}$ with their tails at the same point (fig a).




The angle between their directions is $\phi$; the angle $\phi$ always lies between 0° and 180°. The projection of the vector $\mathbf{B}$ onto the direction of $\mathbf{A}$ is the component of $\mathbf{B}$ parallel to **A** and is equal to $B\cos\phi$.

We define $\mathbf{A}\cdot\mathbf{B}$ to be the magnitude of $\mathbf{A}$ multiplied by the component of $\mathbf{B}$ parallel to $\mathbf{A}$ (fig b).

$$\mathbf{A}\cdot\mathbf{B} = AB\cos\phi$$

Alternatively, we can define $\mathbf{A}\cdot\mathbf{B}$, to be the magnitude of $\mathbf{B}$ multiplied by the component of $\mathbf{A}$ parallel to $\mathbf{B}$, as in fig (c). Hence

$$\mathbf{A}.\mathbf{B} = B(A\cos\phi) = AB\cos\phi$$

The scalar product is a scalar quantity; it is positive when $\phi$ is between 0° and 90°, negative when $\phi$ is between 90° and 180°. When $\phi$ = 90°, $\mathbf{A}\cdot\mathbf{B} = 0$. *The scalar product of two perpendicular vectors is zero.*

For any two vectors A and B,

$$\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A} = AB\cos\phi$$

Thus, the scalar product obeys the commutative law of multiplication; the order of the two vectors does not matter.

The dot product is simply a number which is the product of the magnitudes of the two vectors and the cosine of the angle between them. The number will be positive if $\phi < 90^\circ$, negative if $\phi > 90^\circ$, and zero if $\phi = 90^\circ$.

The special case of the dot product of a vector with itself gives the square of the magnitude of the vector:

$$\mathbf{A}\cdot\mathbf{A} = AA\cos 0^\circ = A^2$$

We now look at the scalar product of the unit vectors. Since $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are all perpendicular to each other, we find

$$\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1\times1\times\cos 0 = 1$$
$$\mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = 1\times1\times\cos 90^\circ = 0$$

Now we express **A** and **B** in terms of their components, expand the product, and use these products of unit vectors:

$$\begin{aligned}\mathbf{A}.\mathbf{B} &= (A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k})\cdot(B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}) \\ &= A_xB_x + A_yB_y + A_zB_z\end{aligned}$$

Thus the scalar product of two vectors is the sum of the products of their respective components.

We note that the components of a vector are equal to the dot product of the vector and the corresponding unit vector. For instance, if

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k},$$

then

$$\begin{aligned}\mathbf{i}\cdot\mathbf{A} &= \mathbf{i}\cdot(A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}) \\ &= A_x\mathbf{i}\cdot\mathbf{i} + A_y\mathbf{i}\cdot\mathbf{j} + A_z\mathbf{i}\cdot\mathbf{k} = A_x\times1 + A_y\times0 + A_z\times0 = A_x\end{aligned},$$

$$\begin{aligned}\mathbf{j}\cdot\mathbf{A} &= \mathbf{j}\cdot(A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}) \\ &= A_x\mathbf{j}\cdot\mathbf{i} + A_y\mathbf{j}\cdot\mathbf{j} + A_z\mathbf{j}\cdot\mathbf{k} = A_x\times0 + A_y\times1 + A_z\times0 = A_y\end{aligned},$$

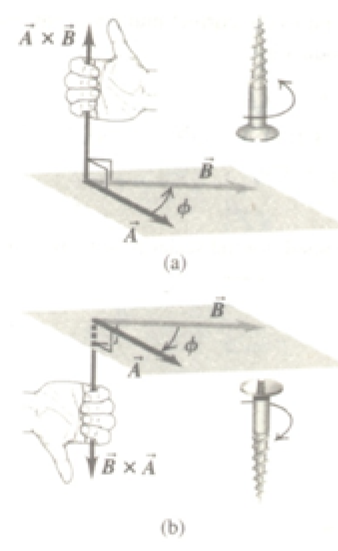
and

$$\begin{aligned}\mathbf{k}\cdot\mathbf{A} &= \mathbf{k}\cdot(A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}) \\ &= A_x\mathbf{k}\cdot\mathbf{i} + A_y\mathbf{k}\cdot\mathbf{j} + A_z\mathbf{k}\cdot\mathbf{k} = A_x\times0 + A_y\times0 + A_z\times1 = A_z\end{aligned}$$

## 2.5

## Cross Product

The **vector product** of two vectors **A** and **B**, also called the **cross product**, is denoted by $\mathbf{A}\times\mathbf{B}$. To define the vector product $\mathbf{A}\times\mathbf{B}$ of two vectors **A** and **B**, We draw the two vectors with their tails at the same point.



The two vectors then lie in a plane. We define the vector product to be a vector quantity with a direction perpendicular to this plane and a magnitude equal to $AB\sin\phi$. That is, if $\mathbf{C}=\mathbf{A}\times\mathbf{B}$, then

$$C = AB\sin\phi$$

We measure the angle $\phi$ from $\mathbf{A}$ toward $\mathbf{B}$ and take it to be the smaller of the two possible angles, so $\phi$ ranges from 0° to 180°. Thus $C$ in the above equation is always positive, as a vector magnitude should be. When **A** and **B** are parallel or antiparallel, $\phi$ = 0° or 180° and C = 0. That is, *the vector product of two parallel or antiparallel vectors is zero.*

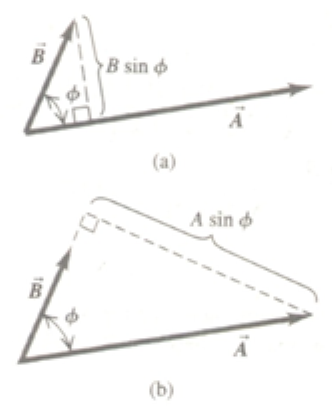
There are always two directions perpendicular to a given plane, one on each side of the plane. We choose one of these as the direction of $\mathbf{A}\times\mathbf{B}$ as follows.

Imagine rotating vector $\mathbf{A}$ about the perpendicular line until it is aligned with $\mathbf{B}$, choosing the smaller of the two possible angles between $\mathbf{A}$ and $\mathbf{B}$. Curl the fingers of your right hand around the perpendicular line so that the fingertips point in the direction of rotation; your thumb will then point in the direction of $\mathbf{A}\times\mathbf{B}$. This right-hand rule is shown in the figure (a). The direction of the vector product is also the direction in which a right-hand screw advances if turned from **A** toward **B**, as shown.

Similarly, we determine the direction of $\mathbf{B}\times\mathbf{A}$ by rotating $\mathbf{B}$ into $\mathbf{A}$ in figure (b) above. The result is a vector that is *opposite* to the vector $\mathbf{A}\times\mathbf{B}$. The vector product is not commutative. For any two vectors $\mathbf{A}$ and $\mathbf{B}$

$$\mathbf{A}\times\mathbf{B}=-\mathbf{B}\times\mathbf{A}$$

There is a geometrical interpretation of the magnitude of the vector product.



In figure (a), $B\sin\phi$ is the component of the vector **B** that is perpendicular to the direction of the vector **A**. The magnitude of $\mathbf{A}\times\mathbf{B}$ equals the magnitude of **A** multiplied by the component of **B** perpendicular to **A**. Figure (b) shows that the magnitude of $\mathbf{A}\times\mathbf{B}$ also equals the magnitude of **B** multiplied by the component of **A** perpendicular to **B**.

The vector product of any vector with itself is zero, so

$$\mathbf{i}\times\mathbf{i}=\mathbf{j}\times\mathbf{j}=\mathbf{k}\times\mathbf{k}=0$$

Each product is a zero *vector* – that is, one with all components equal to zero and an undefined direction. Using the right-hand rule, we find

$$\mathbf{i}\times\mathbf{j}=-\mathbf{j}\times\mathbf{i}=\mathbf{k}$$

$$\mathbf{j}\times\mathbf{k}=-\mathbf{k}\times\mathbf{j}=\mathbf{i}$$

$$\mathbf{k}\times\mathbf{i}=-\mathbf{i}\times\mathbf{k}=\mathbf{j}$$

We express **A** and **B** in terms of their components and the corresponding unit vectors, and we expand the expression for the vector product:

$$\begin{aligned}\mathbf{A}\times\mathbf{B}&=(A_x\hat{i}+A_y\hat{j}+A_z\hat{k})\times(B_x\hat{i}+B_y\hat{j}+B_z\hat{k})\\&=\left(A_yB_z-A_zB_y\right)\hat{i}+\left(A_zB_x-A_xB_z\right)\hat{j}+\left(A_xB_y-A_yB_x\right)\hat{k}\end{aligned}$$

Thus the components of $\mathbf{C}=\mathbf{A}\times\mathbf{B}$ are given by

$$C_x=A_yB_z-A_zB_y$$
$$C_y=A_zB_x-A_xB_z$$
$$C_z=A_xB_y-A_yB_x$$

The vector product can also be expressed in determinant form as

$$\mathbf{A}\times\mathbf{B}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\A_x&A_y&A_z\\B_x&B_y&B_z\end{vmatrix}$$

**Example**: Vector **A** has magnitude 6 units and is in the direction of the +*x*-axis. Vector B has magnitude 4 units and lies in the *xy*-plane, making an angle of 30° with the +*x*-axis. Find the vector product $\mathbf{A}\times\mathbf{B}$.

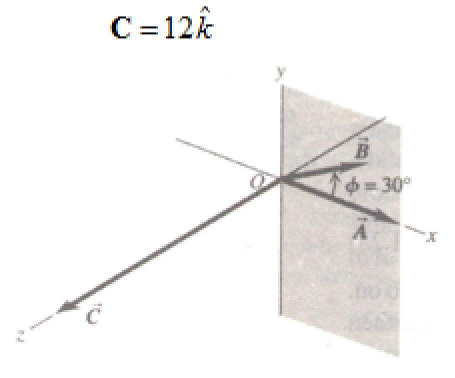
**Solution**:

If $\mathbf{C}=\mathbf{A}\times\mathbf{B}$,

$$C=AB\sin\varphi=6\times4\times\sin30^\circ=12$$

From the right-hand rule, the direction of $\mathbf{A}\times\mathbf{B}$ is along the +*z*-axis, so we have

$$\mathbf{C}=12\hat{k}$$



We can also determine the vector product from the component of the vectors.

$$A_x=6 \qquad A_y=0 \qquad A_z=0$$
$$B_x=4\cos30^\circ=2\sqrt{3} \qquad B_y=4\sin30^\circ=2 \qquad B_z=0$$
$$C_x=0\times0-2\times0=0$$
$$C_y=0\times2\sqrt{3}-6\times0=0$$
$$C_z=6\times2-0\times2\sqrt{3}=12$$

Thus

$$\mathbf{C}=12\hat{k}$$

Module 2.5

**<u>Vector product</u>**

(Home work)

[Work out the following with proper reasoning]

1) If θ is the angle between the vectors $\overline{A}$ and $\overline{B}$, find the magnitude of $(\overline{A}+\overline{B})\times(\overline{A}-\overline{B})$.

**Answer:** $-2AB\sin\theta$

2) If $\overline{A}=A_x\overline{i}+A_y\overline{j}+A_z\overline{k}$ and $\overline{B}=B_x\overline{i}+B_y\overline{j}+B_z\overline{k}$, find the condition for $\overline{A}\parallel\overline{B}$.

**Answer:** $\frac{A_x}{B_x}=\frac{A_y}{B_y}=\frac{A_z}{B_z}$ )

3) If the cross product has the same magnitude as the dot product for two vectors, what is the angle between the vectors?

**Answer:** $45^0$

4) Find $\overline{i}\times\overline{j}\times\overline{k}=?$

**Answer:** Zero

5) If $\overline{A}=3\overline{i}+4\overline{k}$, $\overline{B}=3\overline{k}-7\overline{i}$ and $\overline{C}=2\overline{i}+6\overline{k}$, find the angle between $\overline{A}\times\overline{B}$ and $\overline{C}$.

**Answer:** 90°