10.1. Fundamentals of Matrices

Learning objectives

- To define the matrix, order of a matrix, equality of matrices, negative of a matrix.
- To define the submatrix, transpose, trace of a given matrix
- To study the different types of matrices

AND

- To solve the related problems

Matrices arise naturally in almost all branches of engineering and physical sciences. The rules defining the operations on matrices are usually called matrix algebra. We shall discuss matrix algebra and its use in solving linear system of algebraic equations.

Matrix

An $\boldsymbol{m \times n}$ ***matrix*** is an arrangement of mn elements (not necessarily distinct) in m ***rows*** and n ***columns*** in the form

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad ...(1)$$

We confine to matrices whose elements are real or complex numbers; or real or complex valued functions.

If all the elements of a matrix are real numbers (complex numbers), then it is called a ***real matrix*** (***complex matrix***).

Notice that every element in the matrix is specified by its position in terms of the row and column in which the element is present. For example a_{32} is the element in the third row and second column. In general a_{ij} is the element in i^{th} row and j^{th} column of the matrix.

The matrix (1) is denoted, in compact form, as $A = [a_{ij}]_{m \times n}$, where $i = 1,2, \dots, m$ and $j = 1,2, \dots, n$.

The matrices are usually denoted by upper case letters $A, B, C, \dots, etc$.

Order of a matrix

A matrix A is said be of ***order*** $m \times n$ (read as m by n) if the matrix A has m rows and n columns.

* A matrix $A = [a_{ij}]_{m \times n}$ is said to be a ***rectangular matrix*** if $m \neq n$, i.e., the number of rows in A is not equal to the number of columns in A.
* A matrix $A = [a_{ij}]_{m \times n}$ is said to be a ***square matrix*** if $m = n$, i.e., the number of rows in A is equal to the number of columns in A.
  A matrix of order $n \times n$ is called a ***square matrix of order*** $\boldsymbol{n}$.

Examples

i) $A = \begin{bmatrix} 3 & -2 & 0 \\ \frac{1}{2} & 0 & -6 \end{bmatrix}$ is a real rectangular matrix of order 2×3.

ii) $A = \begin{bmatrix} 1 & w & w^2 \\ w & w^2 & 1 \\ w^2 & 1 & w \end{bmatrix}$ is a complex square matrix of order 3 (where w is a cube root of unity).

iii) $A = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}$ is a real square matrix of order 2.

Equality of matrices

Two matrices A and B are said to be ***equal*** (written as $A = B$) if

(i) A and B are of the same order
(ii) The corresponding elements are equal.

Example: If $A = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 6 \\ 2 & 5 \\ 3 & 4 \end{bmatrix}$ are equal, i.e., $A = B$ iff

$$a = 1, b = 6, c = 2, d = 5, e = 3 \text{ and } f = 4.$$

Null matrix or a zero matrix, Row matrix and column matrix

If each element of a matrix is zero, then it is said to be a ***Null matrix or zero matrix***. The zero matrix of order $m \times n$ is denoted by $O_{m \times n}$ or simply by $\boldsymbol{O}$.

Examples: $O_{2\times3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $O_2 = O_{2\times2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

A matrix is said to be a ***row matrix*** (***row vector***) if it contains only one row.

A matrix is said to be a ***column matrix*** (***column vector***) if it contains only one column.

Examples: $[1 \quad 0 \quad -2]$ is a row matrix (It is a matrix of order 1×3)

$\begin{bmatrix} \frac{2}{3} \\ 1 \\ -1 \end{bmatrix}$ is a column matrix (It is a matrix of order 3×1)

Negative of a Matrix

The matrix obtained from a matrix A by multiplying each element by -1 (i.e., by changing the sign of each element) is called the ***negative of A*** (or ***additive inverse of A***) and it is denoted by $-\boldsymbol{A}$. Thus, if $A = [a_{ij}]_{m \times n}$ then $-A = [-a_{ij}]_{m \times n}$

Example: If $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 4 \\ -6 & -4 & -1 \end{bmatrix}$ then $-A = \begin{bmatrix} -1 & 2 & -3 \\ -2 & -1 & -4 \\ 6 & 4 & 1 \end{bmatrix}$

Submatrix

A ***submatrix*** of a matrix A is an array formed by deleting one or more rows or columns of A. Note that this definition allows the deletion of a combination of rows and columns.

Example: Let $A = \begin{bmatrix} 2 & 5 & -1 \\ 4 & -7 & 13 \\ 3 & -9 & 6 \end{bmatrix}$

Deleting the third row of A gives the submatrix $\begin{bmatrix} 2 & 5 & -1 \\ 4 & -7 & 13 \end{bmatrix}$. Deleting second row and first column of A gives the submatrix $\begin{bmatrix} 5 & -1 \\ 9 & 6 \end{bmatrix}$

Trace of a square matrix

If $A = [a_{ij}]_{n \times n}$ is a square matrix of order n, then the elements $a_{11}, a_{22}, \dots, a_{nn}$ are said to constitute the ***principal diagonal*** (or simply ***diagonal***) of A.

The sum of the diagonal elements of a square matrix A is called the ***trace of A*** and is denoted by $r(A)$, i.e., $Tr(A) = \sum_{i=1}^{n} a_{ii}$

Types of matrices

Diagonal matrix

A square matrix $A = [a_{ij}]_{n \times n}$ is called a ***diagonal matrix*** if $a_{ij} = 0$ for $i \neq j$ (i.e., if each non diagonal element is zero) and is denoted by *diag*$[a_{11}, a_{22}, \dots, a_{nn}]$.

Examples: $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -i \end{bmatrix}$ are diagonal matrices.

Scalar matrix

A square matrix $A = [a_{ij}]_{n \times n}$ is called a ***scalar matrix*** if $a_{ij} = 0$, for $i \neq j$ and $a_{11} = a_{22} = \cdots = a_{nn}$. (i.e., if each non diagonal element is zero and all the diagonal elements are equal to each other)

Examples: $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} -i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & -i \end{bmatrix}$ are scalar matrices

Note that every scalar matrix is a diagonal matrix but not conversely.

Unit matrix or Identity matrix

A square matrix $A = [a_{ij}]_{n \times n}$ is called a ***unit matrix or identity matrix*** if $a_{ij} = 0$, for $i \neq j$ and $a_{11} = a_{22} = \cdots = a_{nn} = 1$ (i.e., if each non diagonal element is zero and each diagonal element is equal to 1)

The identity matrix of order n is usually denoted by $\boldsymbol{I_n}$ or simply $\boldsymbol{I}$ when there is no ambiguity about its order.

$$\boldsymbol{I_2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \boldsymbol{I_3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Triangular matrices

A square matrix $A = [a_{ij}]_{n \times n}$ is said to be an ***upper triangular matrix*** if

$$a_{ij} = 0 \text{ for all } i > j$$

Examples: $\begin{bmatrix} -1 & \frac{2}{5} \\ 0 & 4 \end{bmatrix}, \begin{bmatrix} 2 & -1 & 0 \\ 0 & -2 & 3 \\ 0 & 0 & 4 \end{bmatrix}$ are upper triangular matrices.

A square matrix $A = [a_{ij}]_{n \times n}$ is said to be a ***lower triangular matrix*** if

$$a_{ij} = 0 \text{ for all } i < j$$

Examples: $\begin{bmatrix} 1 & 0 \\ -2 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ are lower triangular matrices.

Note that I_n and O_n (identity and zero matrix of order n) are both upper and lower triangular matrices.

Transpose of a matrix

Let A be a matrix. The matrix obtained by interchanging the row and columns of A is called the ***transpose*** of A. That is, if $A = [a_{ij}]_{n \times n}$ then $A^T = [a_{ji}]_{n \times n}$. The transpose of A is denoted by A^T (or A').

Example: If $A = \begin{bmatrix} 1 & \sqrt{2} & -2 \\ -5 & 2 & 1+\sqrt{7} \end{bmatrix}$ then $A^T = \begin{bmatrix} 1 & -5 \\ \sqrt{2} & 2 \\ -2 & 1+\sqrt{7} \end{bmatrix}$.

Observe that $(A^T)^T = \begin{bmatrix} 1 & \sqrt{2} & -2 \\ -5 & 2 & 1+\sqrt{7} \end{bmatrix} = A$.

Note:

(i) If A is an $m \times n$ matrix then A^T is an $n \times m$ matrix.
(ii) The transpose of a row matrix is a column matrix and the transpose of a column matrix is a row matrix
(iii) For any matrix A, $(A^T)^T = A$.

Symmetric and Skew-symmetric matrices

A square matrix $A = [a_{ij}]_{n \times n}$ is said to be

(i) Symmetric if $A^T = A$, i.e., $a_{ij} = a_{ji}, \forall i, j$
(ii) Skew symmetric if $A^T = -A$, i.e., $a_{ij} = -a_{ji}, \forall i, j$

Note:

(i) The zero matrix O_n is both symmetric and skew symmetric.
(ii) Any diagonal matrix and the identity matrix are symmetric

Example

The matrix $A = \begin{bmatrix} 3 & 5 & -6 \\ 5 & 0 & 2 \\ -6 & 2 & 1 \end{bmatrix}$ is a symmetric matrix for $A^T = \begin{bmatrix} 3 & 5 & -6 \\ 5 & 0 & 2 \\ -6 & 2 & 1 \end{bmatrix} = A$

The matrix $A = \begin{bmatrix} 0 & a & b \\ -a & 0 & -c \\ -b & c & 0 \end{bmatrix}$ is a symmetric matrix for $A^T = \begin{bmatrix} 0 & -a & -b \\ a & 0 & c \\ b & -c & 0 \end{bmatrix} = -A$

The following are simple results

Lemma: If a square matrix of order $\boldsymbol{n}$ is both symmetric and skew symmetric then $\boldsymbol{A}$ is the zero matrix of order $\boldsymbol{n}$.

Proof: We have $A = [a_{ij}]_{n \times n}$ and

A is symmetric $\Longrightarrow a_{ij} = a_{ji}, \forall\, i, j$
A is skew symmetric $\Longrightarrow a_{ij} = -a_{ji}, \forall\, i, j$

Since A is both symmetric and skew symmetric, we have

$$a_{ij} = -a_{ji} = -a_{ij}, \forall i, j$$

i.e., $2a_{ij} = 0, \forall i, j \Rightarrow a_{ij} = 0, \forall\, i, j$

Therefore, A is a zero matrix of order n. Hence the result.

Lemma: If $\boldsymbol{A}$ is a skew symmetric matrix, then all its diagonal elements are zero.

(The diagonal elements of a skew symmetric matrix are all zero).

Proof: We have $A = [a_{ij}]_{n \times n}$ and A is skew symmetric i.e., $a_{ij} = -a_{ji}$ for all i, j.

Taking $i = j$, we get $a_{ii} = -a_{ii}$ for all i, i.e., $2a_{ii} = 0$ for all $i \Rightarrow a_{ii} = 0$, for all i.

Thus, the diagonal elements of a skew symmetric matrix are all zero.

Hence the result.

IP1:

If $\begin{bmatrix} 5x-y & 0 & 1 \\ 2 & 7x-3z & 3 \\ 46-6z & 5 & 12 \end{bmatrix} = \begin{bmatrix} -4z+15 & 0 & 1 \\ 2 & 19-4y & 3 \\ 2x+y & 5 & 12 \end{bmatrix}$, then find the values of $, y, z$.

Solution:

Given that $\begin{bmatrix} 5x-6y & 0 & 1 \\ 2 & 7x-3z & 3 \\ 46-6z & 5 & 12 \end{bmatrix} = \begin{bmatrix} -4z+15 & 0 & 1 \\ 2 & 19-4y & 3 \\ 2x+y & 5 & 12 \end{bmatrix}$

(If two matrices are equal then their corresponding elements are also equal)

$$\therefore \; 5x-6y = -4z+15 \Rightarrow 5x-6y+4z = 15 \qquad \ldots(1)$$

$$7x-3z = 19-4y \Rightarrow 7x+4y-3z = 19 \qquad \ldots(2)$$

$$46-6z = 2x+y \Rightarrow 2x+y+6z = 46 \qquad \ldots(3)$$

$$\left.\begin{array}{l} (2)\times 1 \Rightarrow 7x+4y-3z = 19 \\ (3)\times 4 \Rightarrow 8x+4y+24z = 184 \end{array}\right\} \Rightarrow x+27z = 165$$

$$\left.\begin{array}{l} (1)\times 1 \Rightarrow 5x-6y+4z = 15 \\ (3)\times 6 \Rightarrow 12x+6y+36z = 46 \end{array}\right\} \Rightarrow 17x+40z = 291$$

$$\left.\begin{array}{r} \Rightarrow 17(x+27z) = 165\times 17 \Rightarrow 17x+459z = 2805 \\ \Rightarrow 17x+40z = 291 \end{array}\right\} \Rightarrow z = 6$$

$$\therefore x+27z = 165 \Rightarrow x+162 = 165 \Rightarrow x = 3$$

$$(3) \Rightarrow 2(3)+y+6(6) = 46 \Rightarrow y = 4$$

Therefore, $x = 3, y = 4, z = 6$

IP2.

Consider the following the statements

I. Every diagonal matrix is a scalar matrix

II. An identity matrix is both diagonal and scalar matrix.

A. Only I is true
B. Only II is true
C. Both I and II are true
D. Neither I nor II is true

Answer: **B**

IP3.

Find the additive inverse of the matrix $\boldsymbol{A} = \begin{bmatrix} \boldsymbol{i} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & -\boldsymbol{i} & \mathbf{2} \\ -\mathbf{1} & \mathbf{1} & \mathbf{5} \end{bmatrix}$

Solution:

The additive inverse of A is the negative of A denoted by $-A$ and it is the matrix obtained from the matrix A by changing the sign of each element of A.

Therefore, $-A = \begin{bmatrix} -i & 0 & -1 \\ 0 & i & -2 \\ 1 & -1 & -5 \end{bmatrix}$

IP4.

I. If $A = \begin{bmatrix} -1 & 2 & -3 \\ 2 & 5 & 6 \\ -3 & x & 7 \end{bmatrix}$ is a symmetric matrix, then find x

II. If $A = \begin{bmatrix} 0 & 2 & 1 \\ -2 & 0 & 2 \\ -1 & x & 0 \end{bmatrix}$ is a skew symmetric matrix, then find x

Solution:

I. Given $A = \begin{bmatrix} -1 & 2 & -3 \\ 2 & 5 & 6 \\ -3 & x & 7 \end{bmatrix}$

Therefore, $A^T = \begin{bmatrix} -1 & 2 & -3 \\ 2 & 5 & 6 \\ -3 & x & 7 \end{bmatrix}^T = \begin{bmatrix} -1 & 2 & -3 \\ 2 & 5 & x \\ -3 & 6 & 7 \end{bmatrix}$

Since A is symmetric, $A^T = A$

Therefore, $\begin{bmatrix} -1 & 2 & -3 \\ 2 & 5 & x \\ -3 & 6 & 7 \end{bmatrix} = \begin{bmatrix} -1 & 2 & -3 \\ 2 & 5 & 6 \\ -3 & x & 7 \end{bmatrix}$

$\Rightarrow x = 6$

II. Given $A = \begin{bmatrix} 0 & 2 & 1 \\ -2 & 0 & 2 \\ -1 & x & 0 \end{bmatrix}$

Therefore, $A^T = \begin{bmatrix} 0 & 2 & 1 \\ -2 & 0 & 2 \\ -1 & x & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & -2 & -1 \\ 2 & 0 & x \\ 1 & 2 & 0 \end{bmatrix}$

Since A is skew symmetric, $A^T = -A$

Now, $-A = \begin{bmatrix} 0 & -2 & -1 \\ 2 & 0 & -2 \\ 1 & -x & 0 \end{bmatrix}$

Therefore, $\begin{bmatrix} 0 & -2 & -1 \\ 2 & 0 & x \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 & -1 \\ 2 & 0 & -2 \\ 1 & -x & 0 \end{bmatrix} \Rightarrow x = -2$

P1.

Let p and q be prime numbers.

I. If a matrix has p elements then the possible sizes of the matrix are $1 \times p$ and $p \times 1$

II. If a matrix has pq elements then the possible sizes of the matrix are $1 \times pq$, $p \times q$, $q \times p$ and $pq \times 1$.

A. Only I is true
B. Only II is true
C. Both I and II are true
D. Neither I and II are true

P1.

Let p and q be prime numbers.

I. If a matrix has p elements then the possible sizes of the matrix are $1 \times p$ and $p \times 1$

II. If a matrix has pq elements then the possible sizes of the matrix are $1 \times pq$, $p \times q$, $q \times p$ and $pq \times 1$.

A. Only I is true
B. Only II is true
C. Both I and II are true
D. Neither I and II are true

Answer: **C**

P2.

Match the following

a) Nullmatrix	i) $\begin{bmatrix} 0 & 0 & 0 & -1 \end{bmatrix}$
b) Rowmatrix	ii) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
c) Columnmatrix	iii) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
d) Diagonalmatrix, which is not scalar	iv) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
e) Scalarmatrix$(\neq 1)$	v) $\begin{bmatrix} 2 & -4 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
f) Unitmatrix	vi) $\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$
	vii) $\begin{bmatrix} 5 \\ 7 \\ 2 \end{bmatrix}$
	viii) $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

P2.

Match the following

a) Nullmatrix

b) Rowmatrix

c) Columnmatrix

d) Diagonalmatrix, which is not scalar

e) Scalar matrix ($\neq 1$)

f) Unitmatrix

i) $\begin{bmatrix} 0 & 0 & 0 & -1 \end{bmatrix}$

ii) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

iii) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

iv) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

v) $\begin{bmatrix} 2 & -4 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

vi) $\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$

vii) $\begin{bmatrix} 5 \\ 7 \\ 2 \end{bmatrix}$

viii) $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Ans: $a \rightarrow ii, \quad b \rightarrow i, \quad c \rightarrow vii, \; d \rightarrow iii, \; e \rightarrow viii, f \rightarrow iv,$

P3.

Find the trace of the matrix $A = \begin{bmatrix} -4 & 2 & -\frac{1}{2} \\ 0 & -1 & 2 \\ -\frac{1}{2} & 2 & 1 \end{bmatrix}$

P3.

Find the trace of the matrix $\boldsymbol{A} = \begin{bmatrix} \mathbf{-4} & \mathbf{2} & -\frac{\mathbf{1}}{\mathbf{2}} \\ \mathbf{0} & \mathbf{-1} & \mathbf{2} \\ -\frac{\mathbf{1}}{\mathbf{2}} & \mathbf{2} & \mathbf{1} \end{bmatrix}$

Solution:

The elements of the principal diagonal of A are $-4, -1, 1$. The trace of A is the sum of the elements of the principal diagonal.

Therefore, $Tr(A) = -4 - 1 + 1 = -4$

P4.

Find the number of submatrices of order 2×2 of the matrix $A = \begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix}$.

P4.

Find the number of submatrices of order $\mathbf{2 \times 2}$ of the matrix $\boldsymbol{A} = \begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{c} \\ \boldsymbol{p} & \boldsymbol{q} & \boldsymbol{r} \\ \boldsymbol{x} & \boldsymbol{y} & \boldsymbol{z} \end{bmatrix}$.

Solution:

The 2×2 submatrices of the matrix A arise if we delete one row and one column.

We delete the following rows and columns to get the 2×2 submatrices of the matrix A:

1st row, 1st column ; 1st row, 2nd column ; 1st row, 3rd column;

2nd row, 1st column ; 2nd row, 2nd column ; 2nd row, 3rd column;

3rd row, 1st column ; 3rd row, 2nd column ; 3rd row, 3rd column

Thus, we get 9 submatrices and they are:

$$\begin{bmatrix} q & r \\ y & z \end{bmatrix} \;;\; \begin{bmatrix} p & r \\ x & z \end{bmatrix} \;;\; \begin{bmatrix} p & q \\ x & y \end{bmatrix}$$

$$\begin{bmatrix} b & c \\ y & z \end{bmatrix} \;;\; \begin{bmatrix} a & c \\ x & z \end{bmatrix} \;;\; \begin{bmatrix} a & b \\ x & y \end{bmatrix}$$

$$\begin{bmatrix} b & c \\ q & r \end{bmatrix} \;;\; \begin{bmatrix} a & c \\ p & r \end{bmatrix} \;;\; \begin{bmatrix} a & b \\ p & q \end{bmatrix}$$

10.1. Fundamentals of Matrices

1. Construct a 2×2 matrix whose elements a_{ij} are given by:

   $(i)\ a_{ij} = \frac{(i+j)^2}{2}$ $\quad(ii)\ a_{ij} = \frac{(i-j)^2}{2}$

   $(iii)\ a_{ij} = \frac{(i-2j)^2}{2}$ $\quad(iv)\ a_{ij} = \frac{(2i+j)^2}{2}$

   $(v)\ a_{ij} = \frac{|2i-3j|}{2}$ $\quad(iv)\ a_{ij} = \frac{|-3i+j|}{2}$

2. Construct a 3×4 matrix $A = [a_{ij}]$ whose elements a_{ij} are given by:

   $(i)\ a_{ij} = i + j$ $\quad(ii)\ a_{ij} = i - j$ $\quad(iii)\ a_{ij} = 2i$

   $(iv)\ a_{ij} = j$ $\quad(iv)\ a_{ij} = \frac{1}{2}|-3i + j|$

3. Write the size, rows ,columns and submatrices for the following matrices:

   a) $\begin{bmatrix} 1 & 0 & 2 \\ 3 & 4 & 5 \end{bmatrix}$

   b) $\begin{bmatrix} 8 & 0 \\ 4 & -2 \\ 3 & 6 \end{bmatrix}$

   c) $\begin{bmatrix} 7 & 14 \\ 15 & 14 \end{bmatrix}$

   d) $\begin{bmatrix} 2 & 3 & 4 \\ -3 & 4 & 8 \\ 2 & 3 & 4 \end{bmatrix}$

4. If $\begin{bmatrix} x-1 & 2 & y-5 \\ z & 0 & 2 \\ 1 & -1 & 1+a \end{bmatrix} = \begin{bmatrix} 1-x & 2 & -y \\ 2 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}$ then find the values of x, y, z and a.

5. Find the trace of the matrix $\begin{bmatrix} 1 & 3 & -5 \\ 2 & -1 & 5 \\ 2 & 0 & 1 \end{bmatrix}$

6. Is the matrix $\begin{bmatrix} 0 & 1 & 4 \\ -1 & 0 & 7 \\ -4 & -7 & 0 \end{bmatrix}$ symmetric or skew symmetric?

7. Find x, if the matrix $\begin{bmatrix} -5 & 2 & 3 \\ 2 & x & -2 \\ 3 & -2 & 6 \end{bmatrix}$ is symmetric?

8. Find x, if the following matrix is a skew symmetric.

   $i)\ \begin{bmatrix} 0 & 4 & -2 \\ -4 & 0 & 8 \\ 2 & -8 & x \end{bmatrix}$ $\quad ii)\ \begin{bmatrix} 0 & 1 & 4 \\ -1 & 0 & 7 \\ -x & -7 & 0 \end{bmatrix}$

10.2. Algebra of Matrices

Learning objectives

- To define the Addition, subtraction, Scalar Multiplication, Multiplication of matrices and to study their properties.
- To study the Powers of matrices
  AND
- To solve the related problems.

We confine to matrices whose elements are complex numbers.

Addition of two matrices

If A and B are two matrices of the same order then their *sum* is defined to be the matrix obtained by adding the corresponding elements of A and B and it is denoted by $\boldsymbol{A+B}$.

If $A=[a_{ij}]_{m\times n}$ and $B=[b_{ij}]_{m\times n}$, then

$$A+B=[c_{ij}]_{m\times n}, \text{ where } c_{ij}=a_{ij}+b_{ij} \text{ for all } i,j.$$

Example: If $A=\begin{bmatrix}2&1&-3\\4&1&-2\end{bmatrix}$ and $B=\begin{bmatrix}-\sqrt{2}&1&0\\-1&3+i&2\end{bmatrix}$, then

$$A+B=\begin{bmatrix}2+(-\sqrt{2})&1+1&-3+0\\4+(-1)&1+3+i&-2+2\end{bmatrix}=\begin{bmatrix}2-\sqrt{2}&2&-3\\3&4+i&0\end{bmatrix}$$

Two matrices are said to be *conformable* for addition if they have the same order.

Properties of addition of matrices

Let $A=[a_{ij}], B=[b_{ij}]$ and $C=[c_{ij}]$ be matrices conformable for addition, i.e., they have the same order say $m\times n$. The addition of matrices satisfies the following properties:

i) **Commutative property** : $A+B=B+A$
ii) **Associative property** : $A+(B+C)=(A+B)+C$
iii) **Existence of additive identity:**
There exists $O_{m\times n}$ the zero matrix of order $m\times n$ such that

$$\boldsymbol{A+O_{m\times n}=O_{m\times n}+A=A}$$

The *zero matrix* $O_{m\times n}$ is the *identity for addition* or *additive identity* for addition matrices.

iv) **Existence of additive inverse:**
For each matrix A, there is $-A$ the negative of A such that

$$\boldsymbol{A+(-A)=(-A)+A=0}$$

Here $-A$ is called *additive inverse* of A.

We now study the multiplication of a matrix by a scalar and its properties.

Scalar multiple of a matrix

Let A be a matrix of order $m\times n$ and α be a scalar (i.e., a real or a complex number). The $m\times n$ matrix obtained by multiplying each element of A by α is called *a scalar multiple of A* and is denoted by $\boldsymbol{\alpha A}$.

i.e., if $A=[a_{ij}]_{m\times n}$ then $\alpha A=[\alpha a_{ij}]_{m\times n}$

Example: If $A=\begin{bmatrix}1&-2\sqrt{2}&4\\-7&14&-2\end{bmatrix}$ and $\alpha=\frac{1}{2}$, then

$$\alpha A=\begin{bmatrix}\frac{1}{2}&\frac{1}{2}(-2\sqrt{2})&\frac{4}{2}\\-\frac{7}{2}&\frac{14}{2}&-\frac{2}{2}\end{bmatrix}=\begin{bmatrix}\frac{1}{2}&-\sqrt{2}&2\\-\frac{7}{2}&7&-1\end{bmatrix}$$

Note:

(i) $(-1)A=-A$, for $(-1)A=(-1)[a_{ij}]=[(-1)a_{ij}]=[-a_{ij}]=-A$
(ii) $A-B=A+(-B)=A+(-1)B$.
(iii) $0A=O_{m\times n}$, for $0A=0[a_{ij}]=[0a_{ij}]=[0]_{m\times n}=O_{m\times n}$
(iv) $\alpha O_{m\times n}=O_{m\times n}$, for any scalar α.

Properties of scalar multiplication of a matrix

Let A and B be matrices of the same order and α,β are scalars. Then

(i) $\alpha(A+B)=\alpha A+\alpha B$
(ii) $(\alpha+\beta)A=\alpha A+\beta A$
(iii) $\alpha(\beta A)=(\alpha\beta)A$

Multiplication of matrices

Let A and B be matrices. Matrices A , B are said to be *conformable for multiplication* (in this order) giving the product AB if the number of columns of A is equal to the number rows of B.

Product of two matrices

Let $A=[a_{ik}]_{m\times n}$, $B=[b_{kj}]_{n\times p}$ be two matrices which are conformable for multiplication to give the product AB of order $m\times p$. The *product AB* is defined as

$$AB=C=[c_{ik}]_{m\times p}\,,$$

where $c_{ij}=\sum_{k=1}^{n}a_{ik}b_{kj}$.

Note that c_{ij} is the sum of the products of the elements of i^{th} row of A with the corresponding elements of the j^{th} column of B.

Example: Let $A=\begin{bmatrix}2&-1&-2&1\\-1&1&1&-1\end{bmatrix}$ and $B=\begin{bmatrix}2&-1&0\\0&4&-1\\-2&1&0\\1&3&-2\end{bmatrix}$

Note that A and B are of orders 2×4 and 4×3 respectively. The product AB is defined since A,B are comformable for multiplication (i.e., the number of columns of A is equal to the number of rows of B) and AB is of order 2×3.

$$AB=\begin{bmatrix}2&-1&-2&1\\-1&1&1&-1\end{bmatrix}_{2\times 4}\begin{bmatrix}2&-1&0\\0&4&-1\\-2&1&0\\1&3&-2\end{bmatrix}_{4\times 3}$$

$$=\begin{bmatrix}2(2)+(-1)(0)+(-2)(-2)+1(1)&2(-1)+(-1)(4)+(-2)1+1(3)&2(0)+(-1)(-1)+(-2)(0)+1(-2)\\(-1)(2)+1(0)+1(-2)+(-1)1&(-1)(-1)+1(4)+1(1)+(-1)3&-1(0)+1(-1)+1(0)+(-1)(-2)\end{bmatrix}$$

$$=\begin{bmatrix}9&-5&-1\\-5&3&1\end{bmatrix}_{2\times 3}$$

Note that BA is not defined since the matrices B and A are not conformable for multiplication.

Example: Let $A=\begin{bmatrix}1&2&3\\4&-2&-5\end{bmatrix}$ and $B=\begin{bmatrix}-2&3\\4&-5\\-2&1\end{bmatrix}$

Notice that A and B are of orders 2×3 and 3×2 respectively. The product AB is defined since A,B are conformable for multiplication and the order of AB is 2×2.

The product BA is also defined since B,A are conformable for multiplication and the order of BA is 3×3. Now,

$$AB=\begin{bmatrix}1&2&3\\4&-2&-5\end{bmatrix}_{2\times 3}\begin{bmatrix}-2&3\\4&-5\\-2&1\end{bmatrix}_{3\times 2}$$

$$=\begin{bmatrix}1(-2)+2(4)+3(-2)&1(3)+2(-5)+3(1)\\4(-2)+(-2)(4)+(-5)(-2)&4(3)+(-2)(-5)+(-5)(1)\end{bmatrix}=\begin{bmatrix}0&-4\\-6&17\end{bmatrix}_{2\times 2}$$

$$BA=\begin{bmatrix}-2&3\\4&-5\\-2&1\end{bmatrix}_{3\times 2}\begin{bmatrix}1&2&3\\4&-2&-5\end{bmatrix}_{2\times 3}$$

$$=\begin{bmatrix}(-2)1+3(4)&(-2)2+3(-2)&(-2)3+3(-5)\\4(1)+(-5)4&4(2)+(-5)(-2)&4(3)+(-5)(-5)\\(-2)1+1(4)&(-2)(2)+1(-2)&(-2)3+1(-5)\end{bmatrix}=\begin{bmatrix}10&-10&-21\\-16&18&37\\2&-6&-11\end{bmatrix}_{3\times 3}$$

Observe that AB and BA are defined but $AB\neq BA$.

Note 1: The matrix multiplication, in general, is not commutative.

Note 2: It is known that, for non zero real number a,b we have $ab\neq 0$. But in matrix algebra, the product of two non-zero matrices can be a zero matrix. That is, if $A\neq O$, $B\neq O$ but $AB=O$.

Example: Given an example of two square matrices A and B of the same order for which

(i) $AB=0$ but $BA\neq 0$
(ii) $AB=BA=0$

Solution:

(i) Let $A=\begin{bmatrix}1&0\\0&0\end{bmatrix}$, $B=\begin{bmatrix}0&0\\4&3\end{bmatrix}$. Then

$$AB=\begin{bmatrix}0&0\\0&0\end{bmatrix} \text{ but } BA=\begin{bmatrix}0&0\\4&0\end{bmatrix}\neq 0. \quad \text{(do it!)}$$

(ii) Let $A=\begin{bmatrix}0&3&2\\-3&0&1\\-2&-1&0\end{bmatrix}$, $B=\begin{bmatrix}1&-2&3\\-2&4&-6\\3&-6&9\end{bmatrix}$. Then

$$AB=BA=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}=O \quad \text{(verify!)}$$

Note 3: It is known that for non-zero numbers a,b,c, $a\neq 0$, if $ab=ac$ then $b=c$. That is cancellation laws hold. But in matrix algebra, if $AB=AC$ then B need not be equal to C even if $A\neq 0$.

Example: Let $A=\begin{bmatrix}1&2&0\\-1&1&0\\1&-4&0\end{bmatrix}$, $B=\begin{bmatrix}1&2&3\\-1&1&-1\\-3&4&5\end{bmatrix}$, $C=\begin{bmatrix}1&2&3\\-1&1&-1\\-6&7&-8\end{bmatrix}$

Then $AB=\begin{bmatrix}-1&4&1\\-2&-1&-4\\5&-2&7\end{bmatrix}$ and $AC=\begin{bmatrix}-1&4&1\\-2&-1&-4\\5&-2&7\end{bmatrix}$ (do it!)

Note that, $A\neq O$, $B\neq C$ but $AB=AC$.

Remark: We cannot cancel A in the matrix equation $AB=AC$ even if $A\neq 0$. However, there is an important special case when the cancellation is possible. We will learn this in a later lesson.

Properties of multiplication of matrices

We state the properties of multiplication of matrices without proof.

1. **The associative law**
   The multiplication of matrices is associative, i.e., if the A,B and C are conformable for multiplication then
   $$A(BC)=(AB)C$$
2. **The distributive laws**
   For any matrices A,B and C (subject to the conformability)
   (i) **Left distributive law:** $A(B+C)=AB+AC$
   (ii) **Right distributive law:** $(A+B)C=AC+BC$
3. If A is a matrix of order $m\times n$ then
   $$I_mA=AI_m=A$$

Note: i) If A is a square matrix then $IA=AI=A$. The matrix I is called the *multiplicative identity matrix.*

ii) If A,B are compatible for multiplication and α,β are scalars, then

$$(\alpha A)(\beta B)=(\alpha\beta)(AB)=((\alpha\beta)A)B=A((\alpha\beta)B)$$

Powers of matrices

If A is a square matrix and n is a positive integer, then A multiplied by itself n times is denoted by A^n, i.e.,

$$A^n=\underbrace{A\,A\,A\ldots A}_{n\ times}$$

Further, by definition $A^0=I_n$

Familiar rules of exponents of real numbers hold for matrices.

Lemma: If $\boldsymbol{A}$ is a square matrix of order $\boldsymbol{n}$ and $\boldsymbol{r,s}$ are non negative integers and $\boldsymbol{\alpha}$ is a scalar, then

(i) $\boldsymbol{A^rA^s=A^{r+s}}$ (ii) $\boldsymbol{(A^r)^s=A^{rs}}$ (iii) $\boldsymbol{(\alpha A)^r=\alpha^rA^r}$

Example: If $A=\begin{bmatrix}1&-2\\-1&0\end{bmatrix}$, then compute A^4.

Solution: We have $A^4=A\,A\,A\,A$. However, the number of multiplications can be reduced if we write $A^4=(A^2)^2$. Now,

$$A^2=A\,A=\begin{bmatrix}1&-2\\-1&0\end{bmatrix}\begin{bmatrix}1&-2\\-1&0\end{bmatrix}=\begin{bmatrix}3&-2\\-1&2\end{bmatrix}$$

and $A^4=(A^2)^2=\begin{bmatrix}3&-2\\-1&2\end{bmatrix}\begin{bmatrix}3&-2\\-1&2\end{bmatrix}=\begin{bmatrix}11&-10\\-5&6\end{bmatrix}$

Example: If $A=\begin{bmatrix}3&-4\\1&-1\end{bmatrix}$, then prove that $A^n=\begin{bmatrix}1+2n&-4n\\n&1-2n\end{bmatrix}$ for all positive integers n.

Solution: We prove this by induction.

We have the statement $P(n):A^n=\begin{bmatrix}1+2n&-4n\\n&1-2n\end{bmatrix}$

Taking $n=1$, $\begin{bmatrix}1+2&-4\\1&1-2\end{bmatrix}=\begin{bmatrix}3&-4\\1&-1\end{bmatrix}=A=A^1$

Thus, $P(n)$ is true when $n=1$.

Suppose that $P(n)$ is true for $n=k,(k\geq 1)$ i.e., $A^k=\begin{bmatrix}1+2k&-4k\\k&1-2k\end{bmatrix}$. Then

$$A^{k+1}=AA^k=\begin{bmatrix}3&-4\\1&-1\end{bmatrix}\begin{bmatrix}1+2k&-4k\\k&1-2k\end{bmatrix}$$

$$=\begin{bmatrix}3(1+2k)-4k&3(-4k)-4(1-2k)\\1+2k-k&-4k-1(1-2k)\end{bmatrix}=\begin{bmatrix}3+2k&-4k-4\\1+k&-2k-1\end{bmatrix}$$

$$=\begin{bmatrix}1+2(k+1)&-4(k+1)\\k+1&1-2(k+1)\end{bmatrix}$$

This shows that $P(n)$ is true when $n=k+1$.

By mathematical induction

$$A^n=\begin{bmatrix}1+2n&-4n\\n&1-2n\end{bmatrix}$$

for all positive integers n.

Example: If $\boldsymbol{A=\begin{bmatrix}1&-2&1\\0&1&-1\\3&-1&1\end{bmatrix}}$ then show that $\boldsymbol{A^3-3A^2-A-3I=O}$

Solution: Given Matrix is $A=\begin{bmatrix}1&-2&1\\0&1&-1\\3&-1&1\end{bmatrix}$

$$A^2=A.A=\begin{bmatrix}1&-2&1\\0&1&-1\\3&-1&1\end{bmatrix}\begin{bmatrix}1&-2&1\\0&1&-1\\3&-1&1\end{bmatrix}=\begin{bmatrix}4&-5&4\\-3&2&-2\\6&-8&5\end{bmatrix}$$

$$A^3=A^2.A=\begin{bmatrix}4&-5&4\\-3&2&-2\\6&-8&5\end{bmatrix}\begin{bmatrix}1&-2&1\\0&1&-1\\3&-1&1\end{bmatrix}=\begin{bmatrix}16&-17&13\\-9&10&-7\\21&-25&19\end{bmatrix}$$

Now, A^3-3A^2-A-3I

$$=\begin{bmatrix}16&-17&13\\-9&10&-7\\21&-25&19\end{bmatrix}-3\begin{bmatrix}4&-5&4\\-3&2&-2\\6&-8&5\end{bmatrix}-\begin{bmatrix}1&-2&1\\0&1&-1\\3&-1&1\end{bmatrix}-3\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$=\begin{bmatrix}16&-17&13\\-9&10&-7\\21&-25&19\end{bmatrix}-\begin{bmatrix}12&-15&12\\-9&6&-6\\18&-24&15\end{bmatrix}-\begin{bmatrix}1&-2&1\\0&1&-1\\3&-1&1\end{bmatrix}-\begin{bmatrix}3&0&0\\0&3&0\\0&0&3\end{bmatrix}$$

$$=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}=O$$

Hence $A^3-3A^2-A-3\boldsymbol{I}=O$

P1.

If $\boldsymbol{A} = \begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}$, $\boldsymbol{B} = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix}$ and $\boldsymbol{C} = \begin{bmatrix} 4 & 1 & 2 \\ 0 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ then verify

$\boldsymbol{A + (B - C) = (A + B) - C}$

Solution:

Given $A = \begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}, B = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix}, C = \begin{bmatrix} 4 & 1 & 2 \\ 0 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

Now,

$$B - C = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 4 & 1 & 2 \\ 0 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 3-4 & -1-1 & 2-2 \\ 4-0 & 2-3 & 5-2 \\ 2-1 & 0-2 & 3-3 \end{bmatrix} = \begin{bmatrix} -1 & -2 & 0 \\ 4 & -1 & 3 \\ 1 & 2 & 0 \end{bmatrix}$$

$$A + (B - C) = \begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix} + \begin{bmatrix} -1 & -2 & 0 \\ 4 & -1 & 3 \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1-1 & 2-2 & -3+0 \\ 5+4 & 0-1 & 2+3 \\ 1+1 & -1+2 & 1+0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & -3 \\ 9 & -1 & 5 \\ 2 & 1 & 1 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix} + \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1+2 & 2-1 & -3+2 \\ 5+4 & 0+2 & 2+5 \\ 1+2 & -1+0 & 1+3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 1 & -1 \\ 9 & 2 & 7 \\ 3 & -1 & 4 \end{bmatrix}$$

$$(A + B) - C = \begin{bmatrix} 4 & 1 & -1 \\ 9 & 2 & 7 \\ 3 & -1 & 4 \end{bmatrix} - \begin{bmatrix} 4 & 1 & 2 \\ 0 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -3 \\ 9 & -1 & 5 \\ 2 & 1 & 1 \end{bmatrix}$$

Hence $A + (B - C) = (A + B) - C$

P2.

State and prove the properties of addition of matrices.

Solution:

Properties of addition of matrices

Let $A = [a_{ij}], B = [b_{ij}]$ and $C = [c_{ij}]$ be matrices conformable for addition, i.e., they have the same order say $m \times n$. The addition of matrices satisfies the following properties.

i) **Commutative property :** $A + B = B + A$
ii) **Associative property :** $A + (B + C) = (A + B) + C$
iii) **Existence of additive identity:**
There exists $O_{m\times n}$ zero matrix of order $m \times n$ such that
$$A + O_{m\times n} = O_{m\times n} + A = A$$
The zero matrix $O_{m\times n}$ is the identity for addition or additive identity for addition matrices.
iv) **Existence of additive inverse:**
For each matrix A, there is $-A$ the negative of A such that
$$A + (-A) = (-A) + A = O.$$
Here $-A$ is called additive inverse of A.

Proof:

(i) $A + B = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}] = [b_{ij} + a_{ij}]$

(since the addition of complex numbers is commutative)

$= [b_{ij}] + [a_{ij}] = B + A$

Thus, Addition of matrices is commutative

(ii) $A + (B + C) = [a_{ij}] + [b_{ij} + c_{ij}] = [a_{ij} + (b_{ij} + c_{ij})] = [(a_{ij} + b_{ij}) + c_{ij}]$

(since the addition of complex numbers is associative)

$= [a_{ij} + b_{ij}] + [c_{ij}] = (A + B) + C$

Thus, Addition of matrices is associative

(iii) We have $A = [a_{ij}]_{m\times n}$. There exists an $m \times n$ matrix whose every element is zero. i.e., $O_{m\times n} = [0_{ij}]$ and

$A + O_{m\times n} = [a_{ij} + 0]$ (since 0 is the additive identity in $\boldsymbol{C}$)

$= [a_{ij}] = A$

Similarly, $O_{m\times n} + A = A$. Thus, $A + O_{m\times n} = O_{m\times n} + A = A$

(iv) We have $A = [a_{ij}]_{m\times n}$. For this A there exists $-A = [-a_{ij}]_{m\times n}$, its negative and

$$A + (-A) = [a_{ij} + (-a_{ij})]_{m\times n} = O_{m\times n}$$

Similarly, $(-A) + A = O_{m\times n}$. Thus, $A + (-A) = (-A) + A = O_{m\times n}$

P3.

Find $\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}\begin{bmatrix} -1 & 5 \\ -3 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 & 3 & 0 \\ 4 & 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 3 & -1 \\ 1 & 1 \end{bmatrix}$

Solution: Given

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}_{2\times3}\begin{bmatrix} -1 & 5 \\ -3 & 2 \\ 0 & 3 \end{bmatrix}_{3\times2}\begin{bmatrix} 2 & 1 & 3 & 0 \\ 4 & 1 & 0 & 2 \end{bmatrix}_{2\times4}\begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 3 & -1 \\ 1 & 1 \end{bmatrix}_{4\times2}$$

$$=\begin{bmatrix} -1-9+0 & 5+6+15 \\ -2-12+0 & 10+8+18 \end{bmatrix}_{2\times2}\begin{bmatrix} 2+0+9+0 & 2+2-3+0 \\ 4+0+0+2 & 4+2+0+2 \end{bmatrix}_{2\times2}$$

$$=\begin{bmatrix} -10 & 26 \\ -14 & 36 \end{bmatrix}_{2\times2}\begin{bmatrix} 11 & 1 \\ 6 & 8 \end{bmatrix}_{2\times2}$$

$$=\begin{bmatrix} -110+156 & -10+208 \\ -154+216 & -14+288 \end{bmatrix}_{2\times2}=\begin{bmatrix} 46 & 198 \\ 62 & 274 \end{bmatrix}_{2\times2}$$

$$\therefore \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}\begin{bmatrix} -1 & 5 \\ -3 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 & 3 & 0 \\ 4 & 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 3 & -1 \\ 1 & 1 \end{bmatrix}=\begin{bmatrix} 46 & 198 \\ 62 & 274 \end{bmatrix}$$

P4.

If $\boldsymbol{A} = \begin{bmatrix} \boldsymbol{cos\theta} & \boldsymbol{sin\theta} \\ -\boldsymbol{sin\theta} & \boldsymbol{cos\theta} \end{bmatrix}$, then show that $\boldsymbol{A^n} = \begin{bmatrix} \boldsymbol{cosn\theta} & \boldsymbol{sinn\theta} \\ -\boldsymbol{sinn\theta} & \boldsymbol{cosn\theta} \end{bmatrix}$ for all positive integers.

Solution: We prove this by principle of mathematical induction.

Let the statement $P(n)$ be $A^n = \begin{bmatrix} cosn\theta & sinn\theta \\ -sinn\theta & cosn\theta \end{bmatrix}$

Taking $n = 1$, $A^1 = \begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix} = A$. Thus, $P(n)$ is true for $n = 1$

Suppose that $P(n)$ is true for $n = k, (k \geq 1)$. i.e., $A^k = \begin{bmatrix} cosk\theta & sink\theta \\ -sink\theta & cosk\theta \end{bmatrix}$.

$$A^{k+1} = AA^k = \begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix} \begin{bmatrix} cosk\theta & sink\theta \\ -sink\theta & cosk\theta \end{bmatrix}$$

$$= \begin{bmatrix} cos\theta cosk\theta - sin\theta sink\theta & cos\theta sink\theta + sin\theta cosk\theta \\ -sink\theta cosk\theta - cos\theta sink\theta & -sin\theta sink\theta + cos\theta cosk\theta \end{bmatrix}$$

$$= \begin{bmatrix} cos(k\theta + \theta) & sin(k\theta + \theta) \\ -sin(k\theta + \theta) & cos(k\theta + \theta) \end{bmatrix} = \begin{bmatrix} cos(k+1)\theta & sin(k+1)\theta \\ -sin(k+1)\theta & cos(k+1)\theta \end{bmatrix}$$

Thus, $P(n)$ is true for $n = k + 1$

By the principle of mathematical induction

$$A^n = \begin{bmatrix} cosn\theta & sinn\theta \\ -sinn\theta & cosn\theta \end{bmatrix}$$

is true for all positive integers.

IP1.

If $\boldsymbol{A}$ and $\boldsymbol{B}$ are matrices such that $\boldsymbol{2A-B}=\begin{bmatrix}\boldsymbol{3} & \boldsymbol{-3} & \boldsymbol{0}\\ \boldsymbol{3} & \boldsymbol{3} & \boldsymbol{2}\end{bmatrix}$, $\boldsymbol{2B+A}=\begin{bmatrix}\boldsymbol{4}\\ \boldsymbol{-1}\end{bmatrix}$ then find $\boldsymbol{A}$ and $\boldsymbol{B}$.

Solution:

We have $2A-B=\begin{bmatrix}3 & -3 & 0\\ 3 & 3 & 2\end{bmatrix}$ and $2B+A=A+2B=\begin{bmatrix}4 & 1 & 5\\ -1 & 4 & -4\end{bmatrix}$

Now, $4A-2B=2(2A-B)=2\begin{bmatrix}3 & -3 & 0\\ 3 & 3 & 2\end{bmatrix}=\begin{bmatrix}6 & -6 & 0\\ 6 & 6 & 4\end{bmatrix}$

$$A+2B=\begin{bmatrix}4 & 1 & 5\\ -1 & 4 & -4\end{bmatrix}$$

Adding, we get $5A=\begin{bmatrix}4+6 & 1-6 & 5+0\\ -1+6 & 4+6 & -4+4\end{bmatrix}=\begin{bmatrix}10 & -5 & 5\\ 5 & 10 & 0\end{bmatrix}$

$$\Rightarrow A=\tfrac{1}{5}\begin{bmatrix}10 & -5 & 5\\ 5 & 10 & 0\end{bmatrix}=\begin{bmatrix}2 & -1 & 1\\ 1 & 2 & 0\end{bmatrix}$$

We have $A+2B=\begin{bmatrix}4 & 1 & 5\\ -1 & 4 & -4\end{bmatrix}\Rightarrow 2B=\begin{bmatrix}4 & 1 & 5\\ -1 & 4 & -4\end{bmatrix}-A$

$$\Rightarrow 2B=\begin{bmatrix}4-2 & 1+1 & 5-1\\ -1-1 & 4-2 & -4-0\end{bmatrix}=\begin{bmatrix}2 & 2 & 4\\ -2 & 2 & -4\end{bmatrix}$$

$$\Rightarrow B=\tfrac{1}{2}\begin{bmatrix}2 & 2 & 4\\ -2 & 2 & -4\end{bmatrix}=\begin{bmatrix}1 & 1 & 2\\ -1 & 1 & -2\end{bmatrix}$$

$$\therefore\ A=\begin{bmatrix}2 & -1 & 1\\ 1 & 2 & 0\end{bmatrix}, B=\begin{bmatrix}1 & 1 & 2\\ -1 & 1 & -2\end{bmatrix}$$

IP2.

State and prove the properties of the scalar multiplication of matrices.

Solution:

Properties of scalar multiplication of matrices

Let A and B be matrices of the same order and α, β are scalars. Then

(i) $\alpha(A+B) = \alpha A + \alpha B$
(ii) $(\alpha+\beta)A = \alpha A + \beta A$
(iii) $\alpha(\beta A) = (\alpha\beta)A$

Proof: Let $A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{m\times n}$ and α, β be scalars.

(i) $\alpha(A+B) = \alpha[a_{ij}+b_{ij}] = [\alpha(a_{ij}+b_{ij})] = [\alpha a_{ij} + \alpha b_{ij}]$ (by distributivity)
$= [\alpha a_{ij}] + [\beta b_{ij}] = \alpha A + \alpha B$
(ii) $(\alpha+\beta)A = [(\alpha+\beta)a_{ij}] = [\alpha a_{ij} + \beta a_{ij}]$ (by distributivity)
$= [\alpha a_{ij}] + [\beta a_{ij}] = \alpha A + \beta A$
(iii) $\alpha(\beta A) = \alpha[\beta a_{ij}] = [\alpha(\beta a_{ij})] = [(\alpha\beta)a_{ij}]$ (by associativity for multiplication)
$= (\alpha\beta)[a_{ij}]A = (\alpha\beta)A$

IP3.

If $\boldsymbol{A} = \begin{bmatrix} \mathbf{0} & -\boldsymbol{tan}\frac{\alpha}{2} \\ \boldsymbol{tan}\frac{\alpha}{2} & \mathbf{0} \end{bmatrix}$ **and** $\boldsymbol{I}_{2\times2}$ **is a unit matrix, then show that** $(\boldsymbol{I}-\boldsymbol{A})\begin{bmatrix} \boldsymbol{cos\alpha} & -\boldsymbol{sin\alpha} \\ \boldsymbol{sin\alpha} & \boldsymbol{cos\alpha} \end{bmatrix} = \boldsymbol{I}+\boldsymbol{A}$

Solution:

Given $A = \begin{bmatrix} 0 & -tan\frac{\alpha}{2} \\ tan\frac{\alpha}{2} & 0 \end{bmatrix}$ and $I = A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Now, $(I-A)\begin{bmatrix} cos\alpha & -sin\alpha \\ sin\alpha & cos\alpha \end{bmatrix} = \begin{bmatrix} 1 & tan\frac{\alpha}{2} \\ -tan\frac{\alpha}{2} & 1 \end{bmatrix}\begin{bmatrix} cos\alpha & -sin\alpha \\ sin\alpha & cos\alpha \end{bmatrix}$

$$= \begin{bmatrix} 1 & tan\frac{\alpha}{2} \\ -tan\frac{\alpha}{2} & 1 \end{bmatrix}\begin{bmatrix} \frac{1-tan^2\frac{\alpha}{2}}{1+tan^2\frac{\alpha}{2}} & -\frac{2tan\frac{\alpha}{2}}{1+tan^2\frac{\alpha}{2}} \\ \frac{2tan\frac{\alpha}{2}}{1+tan^2\frac{\alpha}{2}} & \frac{1-tan^2\frac{\alpha}{2}}{1+tan^2\frac{\alpha}{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1-tan^2\frac{\alpha}{2}}{1+tan^2\frac{\alpha}{2}} + \frac{2tan^2\frac{\alpha}{2}}{1+tan^2\frac{\alpha}{2}} & -\frac{2tan\frac{\alpha}{2}}{1+tan^2\frac{\alpha}{2}} + \frac{tan\frac{\alpha}{2}\left(1-tan^2\frac{\alpha}{2}\right)}{1+tan^2\frac{\alpha}{2}} \\ \frac{-tan\frac{\alpha}{2}\left(1-tan^2\frac{\alpha}{2}\right)}{1+tan^2\frac{\alpha}{2}} + \frac{2tan\frac{\alpha}{2}}{1+tan^2\frac{\alpha}{2}} & \frac{2tan^2\frac{\alpha}{2}}{1+tan^2\frac{\alpha}{2}} + \frac{1-tan^2\frac{\alpha}{2}}{1+tan^2\frac{\alpha}{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1+tan^2\frac{\alpha}{2}}{1+tan^2\frac{\alpha}{2}} & -\frac{tan\frac{\alpha}{2}\left(2-1+tan^2\frac{\alpha}{2}\right)}{1+tan^2\frac{\alpha}{2}} \\ \frac{tan\frac{\alpha}{2}\left(-1+tan^2\frac{\alpha}{2}+2\right)}{1+tan^2\frac{\alpha}{2}} & \frac{1+tan^2\frac{\alpha}{2}}{1+tan^2\frac{\alpha}{2}} \end{bmatrix} = \begin{bmatrix} 1 & -tan\frac{\alpha}{2} \\ tan\frac{\alpha}{2} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -tan\frac{\alpha}{2} \\ tan\frac{\alpha}{2} & 0 \end{bmatrix} = \boldsymbol{I}+A$$

IP4.

If $\boldsymbol{A} = \begin{bmatrix} \boldsymbol{i} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{i} \end{bmatrix}$, then prove that $\boldsymbol{A}^{\boldsymbol{n}} = \boldsymbol{I}_2, \boldsymbol{A}, -\boldsymbol{I}_2, -\boldsymbol{A}$ according as $\boldsymbol{n} = 4\boldsymbol{k}$, $4\boldsymbol{k}+1, 4\boldsymbol{k}+2$ and $4\boldsymbol{k}+3$ respectively.

Solution: We have $A = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$ and $A^2 = \begin{bmatrix} i^2 & 0 \\ 0 & i^2 \end{bmatrix}, A^3 = \begin{bmatrix} i^3 & 0 \\ 0 & i^3 \end{bmatrix}, \ldots$

We will prove that $A^n = \begin{bmatrix} i^n & 0 \\ 0 & i^n \end{bmatrix}$ for all positive integers n by induction.

Let the statement $P(n)$ be $A^n = \begin{bmatrix} i^n & 0 \\ 0 & i^n \end{bmatrix}$

Clearly, $P(n)$ is true for $n = 1$

Suppose that $P(n)$ is true for $n = k, (k \geq 1)$. i.e., $A^k = \begin{bmatrix} i^k & 0 \\ 0 & i^k \end{bmatrix}$.

$$A^{k+1} = AA^k = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}\begin{bmatrix} i^k & 0 \\ 0 & i^k \end{bmatrix} = \begin{bmatrix} i^{k+1} & 0 \\ 0 & i^{k+1} \end{bmatrix}$$

Thus, $P(n)$ is true for $n = k+1$

By the principle of mathematical induction $P(n)$ is true for all positive integers n.

If $n = 4k$, then $A^{4k} = \begin{bmatrix} i^{4k} & 0 \\ 0 & i^{4k} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \boldsymbol{I}_2$

If $n = 4k+1$, then $A^{4k+1} = \begin{bmatrix} i^{4k+1} & 0 \\ 0 & i^{4k+1} \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} = A$

If $n = 4k+2$, then $A^{4k+2} = \begin{bmatrix} i^{4k+2} & 0 \\ 0 & i^{4k+2} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -\boldsymbol{I}_2$

If $n = 4k+3$, then $A^{4k+3} = \begin{bmatrix} i^{4k+3} & 0 \\ 0 & i^{4k+3} \end{bmatrix} = \begin{bmatrix} -i & 0 \\ 0 & -i \end{bmatrix} = -A$

Hence the result.

10.2. Algebra of Matrices

Exercises

1. Find $\boldsymbol{A+B}$ and $\boldsymbol{A-B}$

i) $A=\begin{bmatrix}1&4&3&6\\2&1&0&2\\1&-1&0&1\end{bmatrix}$, $B=\begin{bmatrix}2&0&1&-2\\1&1&-3&1\\0&1&1&0\end{bmatrix}$

ii) $A=\begin{bmatrix}1&2&4\\3&1&0\\1&1&0\\2&2&4\end{bmatrix}$, $B=\begin{bmatrix}0&2&3\\3&1&1\\0&1&1\\1&3&2\end{bmatrix}$

iii) $A=\begin{bmatrix}1&4&3&6\\0&2&1&4\\0&0&3&1\\0&0&0&2\end{bmatrix}$, $B=\begin{bmatrix}1&0&0&0\\3&1&0&0\\1&2&4&0\\1&1&1&3\end{bmatrix}$

2. Find the following

i) If $A=\begin{bmatrix}1&4&2\\2&1&4\\3&2&2\end{bmatrix}, B=\begin{bmatrix}2&3&-1\\1&2&4\\1&0&3\end{bmatrix}$, then find $13A-15B$

ii) If $A=\begin{bmatrix}-1&-2&3\\1&2&4\\2&-1&3\end{bmatrix}, B=\begin{bmatrix}1&-2&5\\0&-2&2\\1&2&-3\end{bmatrix}$ and $C=\begin{bmatrix}-2&1&2\\1&1&2\\2&0&1\end{bmatrix}$ then find

$2A+22B-222C$.

iii) If $A=\begin{bmatrix}0&1&2\\2&3&4\\4&5&6\end{bmatrix}, B=\begin{bmatrix}-1&2&3\\0&1&0\\0&0&-1\end{bmatrix}$, then find $B-A$ and $4A-5B$

iv) If $A=\begin{pmatrix}1&2\\3&4\end{pmatrix}, B=\begin{pmatrix}3&8\\7&2\end{pmatrix}$ and $2X+A=B$, then find X

3. Find AB and BA

i) $A=\begin{bmatrix}1&4&-2&3\end{bmatrix}, B=\begin{bmatrix}2&1&-1&2\end{bmatrix}^T$

ii) $A=\begin{bmatrix}1&3&-1&2&0\end{bmatrix}, B=\begin{bmatrix}-1&2&13&4&1\end{bmatrix}^T$

iii) $A=\begin{bmatrix}1&2&1&0\\2&1&1&4\\1&0&2&1\\1&1&2&1\end{bmatrix}, B=\begin{bmatrix}3&4\\4&2\\6&-2\\-1&4\end{bmatrix}$

iv) $A=\begin{bmatrix}2&3&1\\4&1&2\\2&2&6\\1&5&2\end{bmatrix}, B=\begin{bmatrix}5&2&3\\2&0&4\\1&4&7\end{bmatrix}$

4. If $A=\begin{bmatrix}1&-2&3\\2&3&-1\\-3&1&2\end{bmatrix}$ and $B=\begin{bmatrix}1&0&2\\0&2&1\\1&2&0\end{bmatrix}$, then examine whether A and B commute w.r.t multiplication of matrices.

5. If $\theta-\varphi=\frac{\pi}{2}$, then show that

$$\begin{bmatrix}cos^2\theta&cos\theta sin\theta\\cos\theta sin\theta&sin^2\theta\end{bmatrix}\begin{bmatrix}cos^2\varphi&cos\varphi sin\varphi\\cos\varphi sin\varphi&sin^2\varphi\end{bmatrix}=O$$

6. If $A=\begin{bmatrix}1&2&2\\2&1&2\\2&2&1\end{bmatrix}$, then show that $A^2-4A-5\boldsymbol{I}=O$

7. If $A=\begin{bmatrix}1&-2&1\\0&1&-1\\3&-1&1\end{bmatrix}$, then find $A^3-3A^2-A-3\boldsymbol{I}$

8. If $A=\begin{bmatrix}\alpha&0\\1&1\end{bmatrix}$ and $B=\begin{bmatrix}1&0\\3&1\end{bmatrix}$, then find the value of α for which $A^2=B$.

9. If $A=\begin{pmatrix}1&0\\\frac{1}{2}&1\end{pmatrix}$, then find the value of A^{100}

10. Under what conditions is the matrix equation $A^2-B^2=(A-B)(A+B)$ is true?

11. If $A=\begin{bmatrix}0&c&-b\\-c&0&a\\b&-a&0\end{bmatrix}$ and $B=\begin{bmatrix}a^2&ab&ac\\ab&b^2&bc\\ac&bc&c^2\end{bmatrix}$ then show that $AB=BA=O$

12. If ω is a complex cube root of unity, show that

$$\left(\begin{bmatrix}1&\omega&\omega^2\\\omega&\omega^2&1\\\omega^2&1&\omega\end{bmatrix}+\begin{bmatrix}\omega&\omega^2&1\\\omega^2&1&\omega\\\omega&\omega^2&1\end{bmatrix}\right)\begin{bmatrix}1\\\omega\\\omega^2\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

13. If A and B are square matrices of the same order, explain, why in general

i) $(A+B)^2\neq A^2+2AB+B^2$

ii) $(A-B)^2\neq A^2-2AB+B^2$

iii) $(A-B)(A+B)\neq A^2-B^2$

10.3. Some Special Matrices

Learning objectives

- To study the properties of
  - i) transposes of matrices
  - ii) Symmetric and Skew symmetric matrices and
  - iii) Traces of matrices.
- To introduce the concepts of Idempotent, Involutary, Nilpotent and Orthogonal matrices.

  AND

- To solve the related problems.

In this module we study the properties of (i) transposes of matrices (ii) Symmetric and Skew symmetric matrices and (iii) Traces of matrices. Further, we introduce the concepts of Idempotent, Involutary, Nilpotent and Orthogonal matrices.

Properties of transposes of matrices

Lemma: If A and B are matrices of suitable orders, then

(i) $(A+B)^T = A^T + B^T$
(ii) $(\alpha A)^T = \alpha A^T$, where α is a scalar
(iii) $(AB)^T = B^T A^T$ (Reversal Law)

Proof:

(i) Let $A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{m\times n}$. Then $A^T = [a_{ji}]_{n\times m}$, $B^T = [b_{ji}]_{n\times m}$

Now, $A + B = [a_{ij}]_{m\times n} + [b_{ij}]_{m\times n} = [c_{ij}]_{m\times n}$, where $c_{ij} = a_{ij} + b_{ij}$

$$(A+B)^T = \left([c_{ij}]_{m\times n}\right)^T = [c_{ji}]_{n\times m} = [a_{ji} + b_{ji}]_{n\times m}$$
$$= [a_{ji}]_{n\times m} + [b_{ji}]_{n\times m} = A^T + B^T$$

(ii) Let $A = [a_{ij}]_{m\times n}$, α be a scalar. Then

$\alpha A = \alpha[a_{ij}]_{m\times n} = [d_{ij}]_{m\times n}$, where $d_{ij} = \alpha a_{ij}$

Now, $(\alpha A)^T = \left([d_{ij}]_{m\times n}\right)^T = [d_{ji}]_{n\times m} = [\alpha a_{ji}]_{n\times m} = \alpha[a_{ji}]_{n\times m} = \alpha A^T$

(iii) Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{n\times p}$. That is A is comformable to B for multiplication. Then AB is of order $m\times p$. Note that B^T, A^T are of orders $p\times n$, $n\times m$ and $B^T A^T$ is of order $p\times m$. Thus, $(AB)^T$ and $B^T A^T$ are the matrices of the same order $p\times m$. We will now show that $(AB)^T = B^T A^T$ by showing the corresponding entries are equal.

Now, $(i,j)^{th}$ entry of $(AB)^T = (j,i)^{th}$ entry of AB

$$= \sum_{k=1}^{n} a_{jk} b_{ki} = \sum_{k=1}^{n} b_{ki} a_{jk} = \sum_{k=1}^{n} b'_{ik} a'_{kj},$$

where $b'_{ik} = (i,k)^{th}$ entry of $B^T = b_{ki}$, $a'_{kj} = (k,j)^{th}$ entry of $A^T = a_{kj}$

$= (i,j)^{th}$ entry of $B^T A^T$

Thus, $(AB)^T = B^T A^T$

The results for the transpose of a sum and a product can be extended to any number of matrices. For example, for three matrices A, B and C

$$(A+B+C)^T = A^T + B^T + C^T$$
$$(ABC)^T = C^T B^T A^T \text{(reversal law)}$$

Symmetric and Skew symmetric matrices

Lemma: If A is a square matrix then

(i) $A + A^T$ is symmetric
(ii) $A - A^T$ is skew symmetric
(iii) AA^T and $A^T A$ are symmetric

Proof:

(i) $(A + A^T)^T = A^T + (A^T)^T = A^T + A \quad (\because (A^T)^T = A)$
$= A + A^T$
$\Rightarrow A + A^T$ is a symmetric matrix

(ii) $(A - A^T)^T = A^T + (-A^T)^T = A^T + (-1)(A^T)^T = A^T - A = -(A - A^T)$
$\Rightarrow A - A^T$ is a symmetric matrix

(iii) $(AA^T)^T = (A^T)^T A^T \quad (\because (AB)^T = B^T A^T)$
$= AA^T$
$\Rightarrow AA^T$ is a symmetric matrix

Similarly, $A^T A$ is a symmetric matrix.

Note: If A is a symmetric matrix (skew symmetric), α is a scalar then αA is also symmetric (skew symmetric).

Lemma:

Every square matrix can be expressed uniquely as the sum of a symmetric matrix and a skew symmetric matrix.

Proof: Let A be any square matrix. Note that

$$A = \tfrac{1}{2}(A + A^T) + \tfrac{1}{2}(A - A^T) = B + C,$$

where $B = \frac{1}{2}(A + A^T)$ and $C = \frac{1}{2}(A - A^T)$

Now, B is symmetric since $A + A^T$ is symmetric and C is skew symmetric since $A - A^T$ is skew symmetric. Thus, every square matrix can be expressed as a sum of a symmetric matrix and a skew symmetric matrix.

To prove the uniqueness, let $A = R + S$, where $R^T = R$ and $S^T = -S$

Now, $A = R + S \Rightarrow A^T = (R+S)^T = R^T + S^T = R - S$. Thus,

$$A = R + S$$
$$A^T = R - S$$

Adding the above and subtracting second from the first, we get $A + A^T = 2R$ and $A - A^T = 2S$. Therefore, $R = \frac{1}{2}(A + A^T)$ and $S = \frac{1}{2}(A - A^T)$

Thus, every square matrix A can be expressed uniquely as the sum of a symmetric matrix and a skew symmetric matrix and

$$A = \tfrac{1}{2}(A + A^T) + \tfrac{1}{2}(A - A^T)$$

Example: If A and B are symmetric matrices, then

(i) $AB + BA$ is symmetric
(ii) $AB - BA$ is skew symmetric

Proof: We have that A and B are symmetric, i.e., $A^T = A$ and $B^T = B$

(i) $(AB + BA)^T = (AB)^T + (BA)^T$
$= B^T A^T + A^T B^T$ (reversal law)
$= BA + AB = AB + BA$
$\Rightarrow AB + BA$ is symmetric

(ii) $(AB - BA)^T = (AB + (-1)BA)^T = (AB)^T + ((-1)BA)^T$
$= (AB)^T + (-1)(BA)^T$
$= B^T A^T - A^T B^T$ (reversal law)
$= BA - AB = -(AB - BA)$
$\Rightarrow AB - BA$ is skew symmetric

Example: Let A and B be symmetric matrices of the same order. Prove that AB is symmetric if and only if $AB = BA$.

Proof: Given that A and B are symmetric matrices of the same order. Therefore, $A^T = A$, $B^T = B$ and AB, BA are defined.

Suppose that AB is symmetric. Then

$$AB = (AB)^T = B^T A^T = BA$$

Thus, if AB is symmetric then $AB = BA$.

Conversely, suppose that $AB = BA$. Then

$(AB)^T = (BA)^T \Rightarrow (AB)^T = A^T B^T = AB \Rightarrow AB$ is symmetric

Thus, if AB is symmetric then $AB = BA$.

Hence the result.

Properties of trace

Let A and B be square matrices of the same order and α be a scalar. Then

(i) $tr(A + B) = tr(A) + tr(B)$
(ii) $tr(\alpha A) = \alpha\, tr(A)$
(iii) $tr(AB) = tr(BA)$
(iv) $tr(A^T) = tr(A)$

We prove the property 3:

Let $A = [a_{ij}]_{n\times n}$, $B = [b_{ij}]_{n\times n}$.

Let $AB = C = [c_{ij}]_{n\times n}$, where $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$

Then, $tr(AB) = \sum_{i=1}^{n} c_{ii} = \sum_{i=1}^{n}\left(\sum_{k=1}^{n} a_{ik} b_{ki}\right)$.

Further, let $BA = D = [d_{ij}]_{n\times n}$ where $d_{ij} = \sum_{k=1}^{n} b_{ik} a_{kj}$

Then, $tr(BA) = \sum_{i=1}^{n} d_{ii} = \sum_{i=1}^{n}\left(\sum_{k=1}^{n} b_{ik} a_{ki}\right)$. Now,

$tr(AB) = \sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik} b_{ki} = \sum_{i=1}^{n}\sum_{k=1}^{n} a_{ki} b_{ik}$ (exchanging the roles of i and j)

$= \sum_{i=1}^{n}\sum_{k=1}^{n} b_{ik} a_{ki} = tr(BA)$

Similarly, the other results can be proved.

Idempotent Matrix

A square matrix A is said to be an ***idempotent matrix*** if $A^2 = A$.

Example: The matrix $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ is idempotent.

Solution: $A^2 = A \cdot A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$

$= \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = A$ (verify!)

Thus, the given matrix A is idempotent.

Example: If A is an idempotent matrix, then show that $B = \boldsymbol{I} - A$ is idempotent and that $AB = BA = O$.

Solution: Given that A is idempotent, i.e., $A^2 = A$ and $B = \boldsymbol{I} - A$

Now, $B^2 = (\boldsymbol{I} - A)^2 = (\boldsymbol{I} - A)(\boldsymbol{I} - A) = \boldsymbol{II} - A\boldsymbol{I} - \boldsymbol{I}A + AA = \boldsymbol{I} - A - A + A^2$

$= \boldsymbol{I} - A - A + A$ ($\because A$ is idempotent)

$= \boldsymbol{I} - A = B$

Thus, $B = \boldsymbol{I} - A$ is idempotent.

Further $AB = A(\boldsymbol{I} - A) = A - A^2 = A - A = O$

$BA = (\boldsymbol{I} - A)A = A - A^2 = A - A = O$

Thus, $AB = BA = O$.

Involutory matrix

A square matrix A is said to be ***involutary matrix*** if $A^2 = \boldsymbol{I}$.

Example: The matrix $A = \begin{bmatrix} 4 & 3 & 3 \\ -1 & 0 & -1 \\ -4 & -4 & -3 \end{bmatrix}$ is involutory.

Solution: $A^2 = A \cdot A = \begin{bmatrix} 4 & 3 & 3 \\ -1 & 0 & -1 \\ -4 & -4 & -3 \end{bmatrix}\begin{bmatrix} 4 & 3 & 3 \\ -1 & 0 & -1 \\ -4 & -4 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (do it!)

Example: Show that a matrix A is involutory if and only if $(\boldsymbol{I} - A)(\boldsymbol{I} + A) = O$.

Solution: Suppose that A is involutory. Then $A^2 = \boldsymbol{I}$

Now $(\boldsymbol{I} - A)(\boldsymbol{I} + A) = \boldsymbol{I} - A + A - A^2 = \boldsymbol{I} - A^2 = \boldsymbol{I} - \boldsymbol{I} = O$

Conversely, suppose that $(\boldsymbol{I} - A)(\boldsymbol{I} + A) = O$.

$\Rightarrow \boldsymbol{I} - A^2 = O \Rightarrow A^2 = \boldsymbol{I} \Rightarrow A$ is involutory.

Nilpotent matrix

Let A be a square matrix. If n is the least positive integer such that $A^n = O$, then A is called the ***nilpotent matrix of index n***.

Example: The matrix $A = \begin{bmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{bmatrix}$ is a nilpotent matrix of index 2.

Solution: $A^2 = A \cdot A = \begin{bmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{bmatrix}\begin{bmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (do it!)

Here 2 is the least positive integer such that $A^2 = O$

Therefore, A is a nilpotent matrix of index 2.

Orthogonal matrix

A real square matrix A of order n is said to be an ***orthogonal matrix*** if $AA^T = \boldsymbol{I}_n$.

Example: The matrix $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ is an orthogonal matrix.

Solution: $AA^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & -\cos\theta\sin\theta + \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \boldsymbol{I}_2$

Thus, A is an orthogonal matrix.

P1.

Express the matrix $\boldsymbol{A} = \begin{bmatrix} \mathbf{3} & \mathbf{2} & \mathbf{3} \\ \mathbf{4} & \mathbf{5} & \mathbf{3} \\ \mathbf{2} & \mathbf{4} & \mathbf{5} \end{bmatrix}$ as the sum of a symmetric matrix and a skew symmetric matrix.

Solution: Given $A = \begin{bmatrix} 3 & 2 & 3 \\ 4 & 5 & 3 \\ 2 & 4 & 5 \end{bmatrix}$.

It is known that every square matrix can be uniquely expressed as the sum of a symmetric matrix B and a skew symmetric matrix C, i.e., $A = B + C$, where $B = \frac{1}{2}(A + A^T)$, $C = \frac{1}{2}(A - A^T)$.

Now, $A^T = \begin{bmatrix} 3 & 4 & 2 \\ 2 & 5 & 4 \\ 3 & 3 & 5 \end{bmatrix}$

$$B = \frac{1}{2}(A + A^T) = \frac{1}{2}\left\{\begin{bmatrix} 3 & 2 & 3 \\ 4 & 5 & 3 \\ 2 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 3 & 4 & 2 \\ 2 & 5 & 4 \\ 3 & 3 & 5 \end{bmatrix}\right\} = \frac{1}{2}\begin{bmatrix} 6 & 6 & 5 \\ 6 & 10 & 7 \\ 5 & 7 & 10 \end{bmatrix}$$

$$C = \frac{1}{2}(A - A^T) = \frac{1}{2}\left\{\begin{bmatrix} 3 & 2 & 3 \\ 4 & 5 & 3 \\ 2 & 4 & 5 \end{bmatrix} - \begin{bmatrix} 3 & 4 & 2 \\ 2 & 5 & 4 \\ 3 & 3 & 5 \end{bmatrix}\right\} = \frac{1}{2}\begin{bmatrix} 0 & -2 & 1 \\ 2 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

$$\therefore A = \frac{1}{2}\begin{bmatrix} 6 & 6 & 5 \\ 6 & 10 & 7 \\ 5 & 7 & 10 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & -2 & 1 \\ 2 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

P2.

If $\boldsymbol{A} = \begin{bmatrix} \mathbf{2} & \mathbf{-2} & \mathbf{-4} \\ \mathbf{-1} & \mathbf{3} & \mathbf{4} \\ \mathbf{1} & \mathbf{-2} & \mathbf{-3} \end{bmatrix}$ **then prove that** $\boldsymbol{A}$ **is an idempotent matrix?**

Solution: Given $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$

Now, $A^2 = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$

$$= \begin{bmatrix} 4+2-4 & -4-6+8 & -8-8+12 \\ -2-3+4 & 2+9-8 & 4+12-12 \\ 2+2-3 & -2-6+6 & -4-8+9 \end{bmatrix} = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = A$$

$\Rightarrow A^2 = A$

$\Rightarrow A$ is an Idempotent Matrix.

P3.

If the matrix $\begin{bmatrix} \boldsymbol{0} & \boldsymbol{2\beta} & \boldsymbol{\gamma} \\ \boldsymbol{\alpha} & \boldsymbol{\beta} & \boldsymbol{-\gamma} \\ \boldsymbol{\alpha} & \boldsymbol{-\beta} & \boldsymbol{\gamma} \end{bmatrix}$ is an orthogonal matrix, then find the values of $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$.

Solution:

Let $A = \begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix}$, $A^T = \begin{bmatrix} 0 & \alpha & \alpha \\ 2\beta & \beta & -\beta \\ \gamma & -\gamma & \gamma \end{bmatrix}$

Since A is orthogonal, $\therefore AA^T = \boldsymbol{I}$

$$\Rightarrow \begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix} \begin{bmatrix} 0 & \alpha & \alpha \\ 2\beta & \beta & -\beta \\ \gamma & -\gamma & \gamma \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4\beta^2 + \gamma^2 & 2\beta^2 - \gamma^2 & -2\beta^2 + \gamma^2 \\ 2\beta^2 - \gamma^2 & \alpha^2 + \beta^2 + \gamma^2 & \alpha^2 - \beta^2 - \gamma^2 \\ -2\beta^2 + \gamma^2 & \alpha^2 - \beta^2 - \gamma^2 & \alpha^2 + \beta^2 + \gamma^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Equating the corresponding elements, we have

$$\left.\begin{matrix} 4\beta^2 + \gamma^2 = 1 \\ 2\beta^2 - \gamma^2 = 0 \end{matrix}\right\} \Rightarrow \beta = \pm \frac{1}{\sqrt{6}}, \gamma = \pm \frac{1}{\sqrt{3}}$$

$$\alpha^2 + \beta^2 + \gamma^2 = 1 \Rightarrow \alpha^2 + \frac{1}{6} + \frac{1}{3} = 1 \Rightarrow \alpha = \pm \frac{1}{\sqrt{2}}$$

Therefore, $\alpha = \pm \frac{1}{\sqrt{2}}$, $\beta = \pm \frac{1}{\sqrt{6}}$, $\gamma = \pm \frac{1}{\sqrt{3}}$

P4.

If the matrix $\boldsymbol{A} = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{3} \\ \mathbf{5} & \mathbf{2} & \mathbf{6} \\ \mathbf{-2} & \mathbf{-1} & \mathbf{-3} \end{bmatrix}$ **is a Nilpotent matrix, then find its index.**

Solution:

Given $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$

Now, $A^2 = AA = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$

$$= \begin{bmatrix} 1+5-6 & 1+2-3 & 3+6-9 \\ 5+10-12 & 5+4-6 & 15+12-18 \\ -2-5+6 & -2-2+3 & -6-6+9 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix}$$

Again $A^3 = A^2A = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$

$$= \begin{bmatrix} 0+0+0 & 0+0+0 & 0+0+0 \\ 3+15-18 & 3+6-9 & 9+18-27 \\ -1-5+6 & -1-2+3 & -3-6+9 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

$\therefore A^3 = O$

Thus, 3 is the least positive integer such that $A^3 = O$.

Therefore, A is the nilpotent matrix of index 3.

IP1.

If $\boldsymbol{A} = \begin{bmatrix} -1 \\ 2 \\ 3 \\ -5 \end{bmatrix}$, $\boldsymbol{B} = \begin{bmatrix} 6 & -2 & 0 & 1 \end{bmatrix}$ then find $\boldsymbol{tr(5AB)}$ and $\boldsymbol{tr(B^T A^T)}$.

Solution: Given $A = \begin{bmatrix} -1 \\ 2 \\ 3 \\ -5 \end{bmatrix}$, $B = \begin{bmatrix} 6 & -2 & 0 & 1 \end{bmatrix}$

Now, $AB = \begin{bmatrix} -1 \\ 2 \\ 3 \\ -5 \end{bmatrix} \begin{bmatrix} 6 & -2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -6 & 2 & 0 & -1 \\ 12 & -4 & 0 & 2 \\ 18 & -6 & 0 & 3 \\ -30 & 10 & 0 & -5 \end{bmatrix}$

$\Rightarrow tr(AB)$ =sum of the diagonal entries= $-6 - 4 + 0 - 5 = -15$

$\Rightarrow tr(AB) = -15$

Now, $tr(5AB) = 5tr(AB) \qquad (\because tr(kA) = ktr(A))$

$= 5(-15) = -75$

$\Rightarrow tr(5AB) = -75$

We have $(AB)^T = B^T A^T$

$\therefore tr(B^T A^T) = tr((AB)^T) \quad (\because tr(A^T) = trA)$

$= tr(AB) = -15$

Therefore, $tr(5AB) = -75 \quad ; \quad tr(B^T A^T) = -15$

IP2.

If $\boldsymbol{A}$ and $\boldsymbol{B}$ are square matrices of order $\boldsymbol{n}$ such that $\boldsymbol{AB} = \boldsymbol{A}$ and $\boldsymbol{BA} = \boldsymbol{B}$ then show that $\boldsymbol{A}$ and $\boldsymbol{B}$ are Idempotent matrices.

Solution:

Given A and B are square matrices of order n such that $AB = A$, $BA = B$.

We have $ABA = (AB)A = AA = A^2$

Again $ABA = A(BA) = AB = A$

Hence $ABA = A^2 = A \Rightarrow A^2 = A$

$\therefore$ A is an Idempotent matrix.

Similarly, $BAB = B(AB) = B(A) = B$

Again $BAB = (BA)B = BB = B^2$

Hence $BAB = B^2 = B \Rightarrow B^2 = B$

$\therefore$ B is an Idempotent matrix.

IP3.

If both $\boldsymbol{A}-\frac{1}{2}\boldsymbol{I}$ and $\boldsymbol{A}+\frac{1}{2}\boldsymbol{I}$ are orthogonal matrices then prove that $\boldsymbol{A}$ is skew symmetric matrix and $\boldsymbol{A}^2=-\frac{3}{4}\boldsymbol{I}$.

Solution:

Given $A-\frac{1}{2}\boldsymbol{I}$ and $A+\frac{1}{2}\boldsymbol{I}$ are orthogonal matrices.

We have A is an orthogonal matrix $\Rightarrow AA^T=\boldsymbol{I}$

$\therefore \left(A-\frac{1}{2}I\right)\left(A-\frac{1}{2}I\right)^T=\boldsymbol{I}$ and $\left(A+\frac{1}{2}I\right)\left(A+\frac{1}{2}I\right)^T=\boldsymbol{I}$

$\Rightarrow \left(A-\frac{1}{2}I\right)\left(A^T-\frac{1}{2}I^T\right)=\boldsymbol{I}$ and $\left(A+\frac{1}{2}I\right)\left(A^T+\frac{1}{2}I^T\right)=\boldsymbol{I}$

$(\because\ (A+B)^T=A^T+B^T)$

$\Rightarrow \left(A-\frac{1}{2}\boldsymbol{I}\right)\left(A^T-\frac{1}{2}\boldsymbol{I}\right)=\boldsymbol{I}$ and $\left(A+\frac{1}{2}\boldsymbol{I}\right)\left(A^T+\frac{1}{2}\boldsymbol{I}\right)=\boldsymbol{I}$

$\Rightarrow AA^T-\frac{1}{2}A\boldsymbol{I}-\frac{1}{2}\boldsymbol{I}A^T+\frac{1}{4}\boldsymbol{I}=\boldsymbol{I}$ and $AA^T+\frac{1}{2}A\boldsymbol{I}+\frac{1}{2}\boldsymbol{I}A^T+\frac{1}{4}\boldsymbol{I}=\boldsymbol{I}$

$\Rightarrow AA^T-\frac{1}{2}\boldsymbol{I}(A+A^T)+\frac{1}{4}\boldsymbol{I}=\boldsymbol{I}$ and $AA^T+\frac{1}{2}\boldsymbol{I}(A+A^T)+\frac{1}{4}\boldsymbol{I}=\boldsymbol{I}$

Both conditions are true only when

$A+A^T=0\Rightarrow A^T=-A$ and

$AA^T+\frac{1}{4}\boldsymbol{I}=\boldsymbol{I}\Rightarrow A(-A)=\frac{3}{4}\boldsymbol{I}\Rightarrow -A^2=\frac{3}{4}\boldsymbol{I}\Rightarrow A^2=-\frac{3}{4}\boldsymbol{I}$

i.e., A is a skew symmetric matrix and $A^2=-\frac{3}{4}\boldsymbol{I}$

Hence proved.

IP4.

Show that the matrix $\boldsymbol{A} = \begin{bmatrix} \mathbf{-5} & \mathbf{-8} & \mathbf{0} \\ \mathbf{3} & \mathbf{5} & \mathbf{0} \\ \mathbf{1} & \mathbf{2} & \mathbf{-1} \end{bmatrix}$ is an involutary matrix.

Solution:

Given $A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$

Now, $A^2 = A.A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$

$$= \begin{bmatrix} 25 - 24 + 0 & 40 - 40 + 0 & 0 + 0 + 0 \\ -15 + 15 + 0 & -24 + 25 + 0 & 0 + 0 + 0 \\ -5 + 6 - 1 & -8 + 10 - 2 & 0 + 0 + 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \boldsymbol{I}$$

$\Longrightarrow A^2 = A$

$\Longrightarrow A$ is an involutary matrix.

10.3. Some Special Matrices

Exercises

1. If A and B are symmetric (skew symmetric) matrices of the same order, then so is $A+B$.
2. If A is symmetric or skew symmetric, then prove that A^2 is symmetric.
3. Let A and B be skew symmetric matrices of the same order then prove that AB is symmetric iff $AB = BA$.
4. Show that $A = \begin{bmatrix} 2 & -2 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$ are idempotent.
5. Show that $A = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$ are involutory.
6. Show that $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ -1 & -2 & -3 \end{bmatrix}$, $B = \begin{bmatrix} ab & b^2 \\ -a^2 & -b^2 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & -2 & -6 \\ -3 & 2 & 9 \\ 2 & 0 & -3 \end{bmatrix}$ are nilpotent matrices and determine their index.
7. Show that $A = \frac{1}{2}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$ and $B = \frac{1}{2}\begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$ are orthogonal matrices.
8. If $A = \begin{bmatrix} 4 & 0 & 6 \\ 5 & 2 & 1 \\ 7 & 8 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 1 \\ 9 & 1 & 2 \\ 0 & 4 & 1 \end{bmatrix}$ then find $tr(A^T + B)$, $tr(B^T A)$, $tr(A^T B)$, $tr(AB)$.

10.4. Determinant of a Matrix

Learning objectives

- To define the minor and cofactor of an element in a matrix
- To define the determinant of a square matrix.
- To study some properties of the determinant of a matrix
  AND
- To solve the related problems.

Consider the following system of two linear equations in two variables x and y:

$$\left.\begin{aligned} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{aligned}\right\} \quad \dots (1)$$

where at least one of c_1 and c_2 is non zero. It is known that the system (1) has a unique solution or no solution according as $a_1b_2 - a_2b_1$ is not zero or zero respectively. That is, $a_1b_2 - a_2b_1$ determines whether the system (1) has a unique solution or no solution and hence $a_1b_2 - a_2b_1$ is called ***the determinant*** of the system (1). Thus we associate the value $a_1b_2 - a_2b_1$ with the matrix $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ and it is called ***the determinant*** of the 2×2 matrix A, denoted by $|A|$ or $\det A$ or $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$.

The determinant of the 1×1 matrix A is defined as the element of A. That is, if $A = [a]_{1\times1}$, then $|A| = a$.

To define the determinant of a 3×3 matrix we need the following concepts:

Minor and cofactor of an element

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$.

The ***minor of*** a_{ij}, denoted by M_{ij}, is defined as the determinant of the 2×2 matrix obtained by deleting the i^{th} row and j^{th} column of A.

The cofactor of a_{ij} is denoted by A_{ij} and $A_{ij} = (-1)^{i+j}M_{ij}$

For example

$M_{23} =$ The minor of $a_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = a_{11}a_{32} - a_{31}a_{12}$

$A_{23} =$ The cofactor of $a_{23} = (-1)^{2+3}M_{23} = (-1)(a_{11}a_{32} - a_{31}a_{12})$

Example:

Find all minors and cofactors of the elements of the 3×3 matrix $A = \begin{bmatrix} 2 & -1 & 4 \\ 4 & -3 & 1 \\ 1 & 2 & 1 \end{bmatrix}$.

Solution:

Minors	Cofactors
$M_{11} = \begin{vmatrix} -3 & 1 \\ 2 & 1 \end{vmatrix} = -3 - 2 = -5$	$A_{11} = (-1)^{1+1}M_{11} = -5$
$M_{12} = \begin{vmatrix} 4 & 1 \\ 1 & 1 \end{vmatrix} = 4 - 1 = 3$	$A_{12} = (-1)^{1+2}M_{12} = -3$
$M_{13} = \begin{vmatrix} 4 & -3 \\ 1 & 2 \end{vmatrix} = 8 + 3 = 11$	$A_{13} = (-1)^{1+3}M_{13} = 11$
$M_{21} = \begin{vmatrix} -1 & 4 \\ 2 & 1 \end{vmatrix} = -1 - 8 = -9$	$A_{21} = (-1)^{2+1}M_{21} = 9$
$M_{22} = \begin{vmatrix} 2 & 4 \\ 1 & 1 \end{vmatrix} = 2 - 4 = -2$	$A_{22} = (-1)^{2+2}M_{22} = -2$
$M_{23} = \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} = 4 + 1 = 5$	$A_{23} = (-1)^{2+3}M_{23} = -5$
$M_{31} = \begin{vmatrix} -1 & 4 \\ -3 & 1 \end{vmatrix} = -1 + 12 = 11$	$A_{31} = (-1)^{3+1}M_{31} = 11$
$M_{32} = \begin{vmatrix} 2 & 4 \\ 4 & 1 \end{vmatrix} = 2 - 16 = -14$	$A_{32} = (-1)^{3+2}M_{32} = 14$
$M_{33} = \begin{vmatrix} 2 & -1 \\ 4 & -3 \end{vmatrix} = -6 + 4 = -2$	$A_{33} = (-1)^{3+3}M_{33} = -2$

Determinant of a 3×3 matrix

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. The sum of the products of the elements of first row and their corresponding cofactors is called ***the determinant of A***, denoted by $\det A$ or $|A|$.

That is, $\det A = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = \sum_{j=1}^{3} a_{1j}A_{1j}$

Note: $\det A = a_{11}(-1)^{1+1}M_{11} + a_{12}(-1)^{1+2}M_{12} + a_{13}(-1)^{1+3}M_{13}$

$= a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$

$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

Extension of the definition of the determinant to square matrices of order n ($n \geq 4$)

We have defined the concept of determinant to square matrices of order n for $n = 1, 2$ and 3.

For $n = 3$, we have $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ and

$$\det A = \sum_{j=1}^{3} a_{1j}A_{1j} \text{, where } A_{1j} \text{ is the cofactor of } a_{1j} \text{ in } A$$

This concept can be extended to square matrices of order n, $n \geq 4$.

Suppose that the definition of the determinant is true for the square matrices of order $n - 1$. That is, if $B = [b_{ij}]_{n-1 \times n-1}$, then $\det B = \sum_{j=1}^{n-1} b_{1j}B_{1j}$, where B_{1j} is the cofactor of b_{1j}.

Let $A = [a_{ij}]_{n \times n}$. Notice that A_{1j} is the cofactor of a_{1j}, which is the determinant of the submatrix of A of order $n - 1$ by deleting the i^{th} row and j^{th} column. By induction hypothesis A_{ij} is known. Therefore, $\det A$ is defined as

$$\det A = \sum_{j=1}^{n} a_{ij}A_{ij} \text{ ,for the square matrix of order } n.$$

This expansion of the determinant is called ***the expansion by cofactors*** along the i^{th} row.

Note (1): Expansion along any row

We can expand the determinant as the sum of the products of the elements of any row and their corresponding cofactors. That is, if $A = [a_{ij}]_{n \times n}$, then

$$\det A = a_{i1}A_{i1} + a_{i2}A_{i2} + \dots + a_{in}A_{in} = \sum_{j=1}^{n} a_{ij}A_{ij}$$

(expansion along i^{th} row)

Note (2): Expansion along any column

We can expand the determinant as the sum of the products of elements of any column and the corresponding cofactors of the elements of the same column. That is, if $A = [a_{ij}]_{n \times n}$, then

$$\det A = a_{1j}A_{1j} + a_{2j}A_{2j} + \dots + a_{nj}A_{nj} = \sum_{i=1}^{n} a_{ij}A_{ij}$$

(expansion along j^{th} column)

Note (3): This expansion has a very important property that the sum of the products of elements of the i^{th} row and the corresponding cofactors of the elements of the r^{th} row ($r \neq i$) is zero. That is, if $A = [a_{ij}]_{n \times n}$, then

$$a_{i1}A_{r1} + a_{i2}A_{r2} + \dots + a_{in}A_{rn} = 0$$

For example, if $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, $i = 1$ and $r = 3$ then

$a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33} = a_{11}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} + a_{12}(-1)\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} + a_{13}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

$= a_{11}(a_{12}a_{23} - a_{22}a_{13}) - a_{12}(a_{11}a_{23} - a_{21}a_{13}) + a_{13}(a_{11}a_{22} - a_{21}a_{12}) = 0$

The same results holds good for the column expansion

$$a_{1j}A_{1r} + a_{2j}A_{2r} + \dots + a_{nj}A_{nr} = 0 \text{, where } r \neq j$$

Note (4): If A and B are square matrices and $A = B$ then $\det A = \det B$.

Example: Find the determinant of $A = \begin{bmatrix} 2 & -1 & 4 \\ 4 & -3 & 1 \\ 1 & 2 & 1 \end{bmatrix}$.

Solution: $\det A =$ The sum of the products of the elements of the first row and their corresponding cofactors.

$= a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$,

where $a_{11} = 2$, $a_{12} = -1$, $a_{13} = 4$, $A_{11} = -5$, $A_{12} = -3$, $A_{13} = 11$

$= 2(-5) + (-1)(-3) + 4(11) = -10 + 3 + 44 = 37$

Example: Find the determinant of $A = \begin{bmatrix} 2 & -1 & 4 \\ 4 & -3 & 1 \\ 1 & 2 & 1 \end{bmatrix}$

(i) by expanding along 3^{rd} row and
(ii) by expanding along 2^{nd} column

Solution:

(i) Expansion of the determinant along 3^{rd} row:

$\det A = a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33}$

$= 1(11) + 2(14) + 1(-2) = 11 + 29 - 2 = 37$

(ii) Expansion of the determinant along 2^{nd} column:

$\det A = a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32}$

$= (-1)(-3) + (-3)(-2) + 2(14) = 3 + 6 + 28 = 37$

Properties of determinants

The properties of determinants of the matrices of order n are given below. These properties are proved for the case of $n = 3$.

Property 1: The determinant of a triangular matrix is the product of its diagonal elements.

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$ (i.e., A is an upper triangular matrix) then expanding the determinant along the first column, we get

$\det A = a_{11}\begin{vmatrix} a_{22} & a_{23} & a_{24} \\ 0 & a_{33} & a_{34} \\ 0 & 0 & a_{44} \end{vmatrix}$

$= a_{11}a_{22}\begin{vmatrix} a_{33} & a_{34} \\ 0 & a_{44} \end{vmatrix}$ again expanding along the first column

$= a_{11}a_{22}a_{33}a_{44}$

Example: The determinant of $A = \begin{bmatrix} -3 & 1 & -2 & 4 \\ 0 & 2 & -3 & 1 \\ 0 & 0 & -1 & -7 \\ 0 & 0 & 0 & 4 \end{bmatrix}$ is the product of the diagonal elements, i.e., $(-3)(2)(-1)(4) = 24$.

Example: The determinant of the identity matrix is 1.

Property 2: If each element of a row (column) of a square matrix is zero, then the determinant of the matrix is zero.

Notice that the determinant of such a matrix is zero if we expand the determinant along the row (column) containing zeros.

Example: The determinant of a $n \times n$ zero matrix is zero.

Property 3: The determinant of a matrix and its transpose are equal i.e., for any square matrix A, $\det A = \det(A^T)$.

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. Then $A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$

Now, $\det A = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$ (expanding along 1^{st} row)

Expanding $\det A^T$ along 1^{st} column, we get

$\det A^T = a_{11}B_{11} + a_{12}B_{12} + a_{13}B_{13}$,

where B_{11}, B_{12}, B_{13} are the cofactors of a_{11}, a_{12}, a_{13} in A^T

$= a_{11}\begin{vmatrix} a_{22} & a_{32} \\ a_{23} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{31} \\ a_{23} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{31} \\ a_{22} & a_{32} \end{vmatrix}$

$= a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = \det A$

Property 4: If A and B are square matrices of the same order, then

$$\det(AB) = (\det A)(\det B)$$

This property can be verified easily if A, B are matrices of order 2 and A, B are matrices of order 3. The proof of this property in general case is beyond the scope of this course.

Example: If A is an orthogonal matrix, then $\det A = \pm 1$

Example: The determinant of a nilpotent matrix is zero.

Property 5: For any positive integer m, $\det(A^m) = (\det A)^m$

Property 6: If each element in a row (column) of a square matrix is the sum of two elements then its determinant can be expressed as the sum of the determinants as shown below:

$$\begin{vmatrix} a_{11} + b_1 & a_{12} & a_{13} \\ a_{21} + b_2 & a_{22} & a_{23} \\ a_{31} + b_3 & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}$$

The above follows easily as shown below:

$\begin{vmatrix} a_{11} + b_1 & a_{12} & a_{13} \\ a_{21} + b_2 & a_{22} & a_{23} \\ a_{31} + b_3 & a_{32} & a_{33} \end{vmatrix}$

$= (a_{11} + b_1)\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - (a_{21} + b_2)\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + (a_{31} + b_3)\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$

$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} + b_1\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

$-b_2\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + b_3\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$

$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}$

Hence the result

Property 7: If the elements of a square matrix are polynomials in x and its determinant is zero when $x = a$, then $x - a$ is a factor of the determinant of the matrix.

Let $A(x) = \begin{bmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{bmatrix}$. Note that $\det[A(x)]$ is a polynomial in x. If $\det[A(a)] = 0$, then $x - a$ is a factor of $\det[A(x)]$ (by remainder theorem).

P1.

Find the determinant of $\boldsymbol{A} = \begin{bmatrix} \mathbf{1} & \mathbf{3} & \mathbf{-3} \\ \mathbf{-3} & \mathbf{-5} & \mathbf{2} \\ \mathbf{-4} & \mathbf{4} & \mathbf{-6} \end{bmatrix}$

Solution: We use the cofactor expansion along the first row.

$$\det A = \sum_{j=1}^{3} a_{1j} A_{1j}$$

$$= 1(-1)^{1+1} \begin{vmatrix} -5 & 2 \\ 4 & -6 \end{vmatrix} + 3(-1)^{1+2} \begin{vmatrix} -3 & 2 \\ -4 & -6 \end{vmatrix} + (-3)(-1)^{1+3} \begin{vmatrix} -3 & -5 \\ -4 & 4 \end{vmatrix}$$

$$= (-5)(-6) - 2(4) - 3[(-3)(-6) - (-4)2] - 3[-3(4) - (-4)(-5)]$$

$$= 22 - 3(26) - 3(-32) = 40$$

P2.

Find the determinant of the matrix $\boldsymbol{C} = \begin{bmatrix} -1 & 0 & -2 & 4 \\ 0 & 1 & -3 & -4 \\ 0 & 3 & -5 & 4 \\ -2 & -3 & 2 & -6 \end{bmatrix}$

Solution: We use the cofactor expansion along the first column. (why?)

$$\det A = \sum_{i=1}^{4} c_{i1} C_{i1}$$

$$= (-1)(-1)^{1+1} \begin{vmatrix} 1 & -3 & -4 \\ 3 & -5 & 4 \\ -3 & 2 & -6 \end{vmatrix} + 0 + 0 - 2(-1)^{1+4} \begin{vmatrix} 0 & -2 & 4 \\ 1 & -3 & -4 \\ 3 & -5 & 4 \end{vmatrix}$$

$= -\det A^T + 2\det B^T$, where $A = \begin{bmatrix} 1 & -3 & -4 \\ 3 & -5 & 4 \\ -3 & 2 & -6 \end{bmatrix}, B = \begin{bmatrix} 0 & -2 & 4 \\ 1 & -3 & -4 \\ 3 & -5 & 4 \end{bmatrix}$

$$= -\det A + 2\det B \; (\because \det X^T = \det X)$$

$= -40 + 2(48) = 56$ (see **P1** and **IP1**)

P3.

Find the determinant of $\boldsymbol{A} = \begin{bmatrix} -1 & 2+i & 3 \\ 1-i & i & 1 \\ 3i & 2 & -1+i \end{bmatrix}$

Solution:

$$\det A = \sum_{j=1}^{3} a_{1j} A_{1j}$$

$$= (-1)(-1)^{1+1}\begin{vmatrix} i & 1 \\ 2 & -1+i \end{vmatrix} + (2+i)(-1)^{1+2}\begin{vmatrix} 1-i & 1 \\ 3i & -1+i \end{vmatrix} + 3(-1)^{1+3}\begin{vmatrix} 1-i & i \\ 3i & 2 \end{vmatrix}$$

$$= (-1)[i(-1+i) - 2] - (2+i)[(1-i)(-1+i) - 3i] + 3[2(1-i) - 3i^2]$$

$$= (-1)[-i - 1 - 2] - (2+i)[-(1-i^2) - 3i] + 3[2 - 2i + 3]$$

$$= i + 3 - (2+i)[-2 - 3i] + 3[5 - 2i]$$

$$= i + 3 + 4 + 2i + 6i - 3 + 15 - 10i = 19 - i$$

P4.

If $\boldsymbol{A}$ is an orthogonal matrix (of order $\boldsymbol{n}$) then prove that $\mathbf{det}\,\boldsymbol{A} = \pm\mathbf{1}$.

Solution:

A is an orthogonal matrix of order $n \Rightarrow AA^T = A^TA = \boldsymbol{I}_n$

Taking determinants on both sides

$\det(AA^T) = \det \boldsymbol{I}_n$

$\Rightarrow \det A \det A^T = 1 \Rightarrow \det A \det A = 1$

$\Rightarrow (\det A)^2 = 1 \Rightarrow \det A = \pm 1$ ($\because$ A is a real matrix)

IP1.

Find the determinant of $\boldsymbol{B} = \begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{3} \\ \mathbf{-2} & \mathbf{-3} & \mathbf{-5} \\ \mathbf{4} & \mathbf{-4} & \mathbf{4} \end{bmatrix}$

Solution: We use the cofactor expansion along the first row

$$\det A = \sum_{j=1}^{3} b_{1j} B_{1j}$$

$$= 0(-1)^{1+1} \begin{vmatrix} -3 & -5 \\ -4 & 4 \end{vmatrix} + 1(-1)^{1+2} \begin{vmatrix} -2 & -5 \\ 4 & 4 \end{vmatrix} + 3(-1)^{1+3} \begin{vmatrix} -2 & -3 \\ 4 & -4 \end{vmatrix}$$

$$= 0 - (-8 + 20) + 3(8 + 12) = -12 + 60 = 48$$

IP2.

Prove that the determinant of an identity matrix of order $\boldsymbol{n}$ is 1.

Solution:

Note that $\boldsymbol{I}_n$ is a triangular matrix with each diagonal element 1.

Therefore $\det \boldsymbol{I}_n$ is the product of diagonal elements. Thus,

$$\det \boldsymbol{I}_n = \underbrace{1 \cdot 1 \cdot 1 \cdot \ldots \cdot 1}_{n\ times} = 1$$

IP3.

Find the determinant of $\boldsymbol{A} = \begin{bmatrix} \boldsymbol{i} & \mathbf{2} & \mathbf{1} \\ \mathbf{3} & \mathbf{1}+\boldsymbol{i} & \mathbf{2} \\ -\mathbf{2}\boldsymbol{i} & \mathbf{1} & \mathbf{4}+\boldsymbol{i} \end{bmatrix}$

Solution:

$$\det A = \sum_{j=1}^{3} a_{1j} A_{1j}$$

$$= (i)(-1)^{1+1} \begin{vmatrix} 1+i & 2 \\ 1 & 4+i \end{vmatrix} + (2)(-1)^{1+2} \begin{vmatrix} 3 & 2 \\ -2i & 4+i \end{vmatrix} + (1)(-1)^{1+3} \begin{vmatrix} 3 & 1+i \\ -2i & 1 \end{vmatrix}$$

$$= i[(1+i)(4+i) - 2] - 2[3(4+i) + 4i] + 1[3 + 2i(1+i)]$$

$$= i[4 + 5i - 1] - 2[12 + 7i] + [1 + 2i]$$

$$= 4i - 5 - i - 24 - 14i + 1 + 2i$$

$$= -9i + 28$$

IP4.

If $\boldsymbol{A}$ is a nilpotent matrix, then prove that $\mathbf{det}\,\boldsymbol{A} = \mathbf{0}$.

Solution:

Let A be a nilpotent matrix of index m , then $A^m = O$.

Then $\det(A^m) = \det O \Rightarrow (\det A)^m = 0 \Rightarrow \det A = 0$

10.4. Determinant of a Matrix

Exercises

Evaluate the determinants of the following matrices

(i) $\begin{bmatrix} -1 & 7 \\ 3 & 8 \end{bmatrix}$

(ii) $\begin{bmatrix} 4 & -5 \\ 2 & 3 \end{bmatrix}$

(iii) $\begin{bmatrix} 2+i & -1+3i \\ 1-2i & 3-i \end{bmatrix}$

(iv) $\begin{bmatrix} 6 & 4i \\ -6i & 2i \end{bmatrix}$

(v) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ along second column

(vi) $\begin{bmatrix} -1 & 3 & 2 \\ 4 & -8 & 1 \\ 2 & 2 & 5 \end{bmatrix}$ along 3rd column

(vii) $\begin{bmatrix} 0 & 1+i & 2 \\ -2i & 0 & 1-i \\ 3 & 4i & 0 \end{bmatrix}$ along 2nd row

(viii) $\begin{bmatrix} i & 2+i & 0 \\ -1 & 3 & 2i \\ 0 & -1 & 1-i \end{bmatrix}$ along 2nd column

(ix) $\begin{bmatrix} 0 & 2 & 1 & 3 \\ 1 & 0 & -2 & 2 \\ 3 & -1 & 0 & 1 \\ -1 & 1 & 2 & 0 \end{bmatrix}$ along 4th column

(x) $\begin{bmatrix} 1 & 0 & -2 & 3 \\ -3 & 1 & 1 & 2 \\ 0 & 4 & -1 & 1 \\ 2 & 3 & 0 & 1 \end{bmatrix}$ along 3rd column

10.5. Evaluation of determinants using properties

Learning objectives:

- To evaluate the determinant by using elementary row (column) operations.
- To study some properties of the determinants.

AND

- To solve the related problems.

Elementary row (column) operations

There are two types of elementary matrix operations- row operations and column operations. These are useful in matrix algebra. (for example in the evaluation of determinants).

Let A be an $m \times n$ matrix. Let $R_1, R_2, \ldots, R_m$ be the rows and $C_1, C_2, \ldots, C_n$ be the columns of A. Any one of the following three operations on the rows (columns) of A is called an elementary row (column) operation.

i) Interchange of any two rows (columns) of A.

Interchange of the rows R_i and R_j (column C_i and C_j) is denoted by $R_i \leftrightarrow R_j$ $(C_i \leftrightarrow C_j)$.

ii) Multiplying the elements of any row (column) of A by a non zero scalar α.

Multiplying the elements of row R_i (column C_j) by a non zero scalar α is denoted by $R_i \to \alpha R_i$ $(C_i \to \alpha C_i)$.

iii) Adding any scalar multiple of the elements of a row (column) to the corresponding elements of another row (column).

$R_i \to R_i + \alpha R_j$ $(C_i \to C_i + \alpha C_j)$ denotes adding α multiples of the elements of R_j (C_j) to the corresponding elements of R_i (C_i).

Property 8: ***If a matrix B is formed from a matrix A by interchanging any two rows (columns) of A, then***

$$\det B = -\det A$$

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. Then

$$\det A = \sum_{j=1}^{3} a_{1j}A_{1j}$$

$$= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

Let B be the matrix obtained from A by interchanging the 1[st] and 3[rd] row (i.e., $R_1 \leftrightarrow R_3$). Then $B = \begin{bmatrix} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \end{bmatrix}$.

Using the cofactor expansion along the 3[rd] row,

$$\det B = a_{11}(-1)^{3+1}\begin{vmatrix} a_{32} & a_{33} \\ a_{22} & a_{23} \end{vmatrix} + a_{12}(-1)^{3+2}\begin{vmatrix} a_{31} & a_{33} \\ a_{21} & a_{23} \end{vmatrix} + a_{13}(-1)^{3+3}\begin{vmatrix} a_{31} & a_{32} \\ a_{21} & a_{22} \end{vmatrix}$$

$$= a_{11}(a_{32}a_{23} - a_{22}a_{33}) - a_{12}(a_{31}a_{23} - a_{21}a_{33}) + a_{13}(a_{31}a_{22} - a_{21}a_{32})$$

$$= (-1)[(a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})]$$

$$= -\det A$$

Property 9: ***If two rows (columns) of a square matrix A are identical, then det A = 0.***

Proof: Let A be a square matrix when i^{th} and j^{th} rows (columns) are identical. Let B be the matrix obtained from A by interchanging i^{th} and j^{th} rows (columns). By property (1) $\det B = -\det A$. Notice that $B = A$ (since the i^{th} and j^{th} rows are identical). Therefore, $\det B = \det A$. Thus,

$$\det B = \det A = -\det A \Rightarrow 2\det A = 0 \Rightarrow \det A = 0.$$

Example: Show that $\begin{vmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$.

Solution:

$$\begin{vmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{vmatrix} = \begin{vmatrix} a & b+c & c+a \\ b & c+a & a+b \\ c & a+b & b+c \end{vmatrix} + \begin{vmatrix} b & b+c & c+a \\ c & c+a & a+b \\ a & a+b & b+c \end{vmatrix} \text{ (by property 6)}$$

$$= \begin{vmatrix} a & b & c+a \\ b & c & a+b \\ c & a & b+c \end{vmatrix} + \begin{vmatrix} a & c & c+a \\ b & a & a+b \\ c & b & b+c \end{vmatrix} + \underbrace{\begin{vmatrix} b & b & c+a \\ c & c & a+b \\ a & a & b+c \end{vmatrix}}_{\text{zero, since two columns are identical}} + \begin{vmatrix} b & c & c+a \\ c & a & a+b \\ a & b & b+c \end{vmatrix}$$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} + \underbrace{\begin{vmatrix} a & b & a \\ b & c & b \\ c & a & c \end{vmatrix}}_{zero} + \underbrace{\begin{vmatrix} a & c & c \\ b & a & a \\ c & b & b \end{vmatrix}}_{zero} + \begin{vmatrix} a & c & a \\ b & a & b \\ c & b & c \end{vmatrix} + \underbrace{\begin{vmatrix} b & b & c \\ c & c & a \\ a & a & b \end{vmatrix}}_{zero} + \begin{vmatrix} b & b & a \\ c & c & b \\ a & a & c \end{vmatrix} + \underbrace{\begin{vmatrix} b & c & c \\ c & a & a \\ a & b & b \end{vmatrix}}_{zero} + \begin{vmatrix} b & c & a \\ c & a & b \\ a & b & c \end{vmatrix}$$

$$= 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \quad \left\{since \begin{vmatrix} b & c & a \\ c & a & b \\ a & b & c \end{vmatrix} = (-1)\begin{vmatrix} b & a & c \\ c & b & a \\ a & c & b \end{vmatrix} = (-1)^2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}\right\}$$

Property 10: ***If a matrix B is obtained from a matrix A by multiplying the elements of a row (column) by a scalar α, then det B = α det A.***

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. Then $\det A = \sum_{j=1}^{3} a_{3j}A_{3j}$. Let B be the matrix obtained from A by multiplying the elements of 3[rd] row by the scalar α (i.e., $R_3 \to \alpha R_3$). Then

$$B = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \alpha a_{31} & \alpha a_{32} & \alpha a_{33} \end{bmatrix}$$

Notice that the cofactor of αa_{3j} in B is same as the cofactor of a_{3j} in A for $j = 1,2,3$. Therefore,

$$\det B = \sum_{j=1}^{3} \alpha a_{3j}(cofactor\ of\ \alpha a_{3j}) = \sum_{j=1}^{3} \alpha a_{3j}A_{3j} = \alpha\sum_{j=1}^{3} a_{3j}A_{3j} = \alpha \det A$$

Property 11: ***If A is a square matrix of order n and α is a scalar then***

$$\boldsymbol{\det(\alpha A) = \alpha^n \det A}$$

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ be a square matrix of order 3 and α be a scalar then

$$\det(\alpha A) = \begin{vmatrix} \alpha a_{11} & \alpha a_{12} & \alpha a_{13} \\ \alpha a_{21} & \alpha a_{22} & \alpha a_{23} \\ \alpha a_{31} & \alpha a_{32} & \alpha a_{33} \end{vmatrix} = \alpha\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ \alpha a_{21} & \alpha a_{22} & \alpha a_{23} \\ \alpha a_{31} & \alpha a_{32} & \alpha a_{33} \end{vmatrix}$$

$$= \alpha\cdot\alpha\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \alpha a_{31} & \alpha a_{32} & \alpha a_{33} \end{vmatrix} = \alpha^2\cdot\alpha\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \alpha^3 \det A$$

Example: If A is skew symmetric matrix of odd order, then $\det A = 0$.

Solution: Let A be a skew symmetric matrix of odd order say $2k+1$. Then $A = -A^T$.

$\Rightarrow \det A = \det(-A^T) = (-1)^{2k+1}\det A^T$ (by property 11)

$= (-1)\det A$ ($\because \det(A^T) = \det A$)

$\Rightarrow 2\det A = 0 \Rightarrow \det A = 0$

Property 12: ***If the corresponding elements of two rows (columns) of a square matrix A are in the same ratio, then the determinant of A is zero.***

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \alpha a_{21} & \alpha a_{22} & \alpha a_{23} \end{bmatrix}$. Then

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \alpha a_{21} & \alpha a_{22} & \alpha a_{23} \end{vmatrix} = \alpha\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{21} & a_{22} & a_{23} \end{vmatrix} \text{ (by property 10)}$$

$= \alpha.0 = 0$ (since the 2[nd] and 3[rd] rows are identical)

Property 13: ***If a matrix B is obtained from a matrix A by adding a scalar multiple of the elements of a row (column) to the corresponding elements of another row (column), then***

$$\boldsymbol{\det B = \det A}$$

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. Let B be the matrix obtained from A by adding α multiples of the elements of 2[nd] row to the corresponding elements of the 3[rd] row (i.e., $R_3 \to R_3 + \alpha R_2$). Then

$$B = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} + \alpha a_{21} & a_{32} + \alpha a_{22} & a_{33} + \alpha a_{23} \end{bmatrix}$$

Now, $\det B = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} + \alpha a_{21} & a_{32} + \alpha a_{22} & a_{33} + \alpha a_{23} \end{vmatrix}$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \alpha a_{21} & \alpha a_{22} & \alpha a_{23} \end{vmatrix} \text{ (by property 6)}$$

$= |A| + 0$ (by property 12)

$= |A|$

Note: The effect of an elementary row (column) operations on the determinant of a matrix A are given in properties 8, 10 and 12.

These properties of the determinants can be used to simplify the evaluation of a determinant.

Example: Show that $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$

Solution:

$\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$ applying $\boldsymbol{R_1 \longrightarrow R_1 + R_2 + R_3}$

$$= \begin{vmatrix} 0 & 0 & 0 \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$$

$= 0$ (since each element in the 1[st] row is zero)

Example: Show that $\begin{vmatrix} 3a & b-a & c-a \\ a-b & 3b & c-b \\ a-c & b-c & 3c \end{vmatrix} = 3(a+b+c)(ab+bc+ca)$

Solution:

$\begin{vmatrix} 3a & b-a & c-a \\ a-b & 3b & c-b \\ a-c & b-c & 3c \end{vmatrix}$ applying $\boldsymbol{C_1 \to C_1 + C_2 + C_3}$

$$= \begin{vmatrix} a+b+c & -a+b & -a+c \\ a+b+c & 3b & -b+c \\ a+b+c & -c+b & 3c \end{vmatrix}$$

$= (a+b+c)\begin{vmatrix} 1 & -a+b & -a+c \\ 1 & 3b & -b+c \\ 1 & -c+b & 3c \end{vmatrix}$ applying $\boldsymbol{R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1}$

$$= (a+b+c)\begin{vmatrix} 1 & b-a & c-a \\ 0 & 2b+a & a-b \\ 0 & a-c & 2c+a \end{vmatrix}$$

$= (a+b+c)[(2b+a)(2c+a) - (a-c)(a-b)]$

(expanding along the first column)

$= (a+b+c)\{(4bc + 2ab + 2ca + a^2) - (bc - ab - ac + a^2\}$

$= (a+b+c)(3bc + 3ab + 3ca) = 3(a+b+c)(bc+ab+ca)$

$$\therefore \begin{vmatrix} 3a & b-a & c-a \\ a-b & 3b & c-b \\ a-c & b-c & 3c \end{vmatrix} = 3(a+b+c)(ab+bc+ca)$$

Example: If the determinant value of the matrix $A = \begin{bmatrix} 1 & -1 & 0 & x \\ 3 & -2 & 5 & 6 \\ 4 & 1 & 7 & 3 \\ 0 & 1 & 2 & 3 \end{bmatrix}$ is -69 then find the value of x.

Solution: $|A| = \begin{vmatrix} 1 & -1 & 0 & x \\ 3 & -2 & 5 & 6 \\ 4 & 1 & 7 & 3 \\ 0 & 1 & 2 & 3 \end{vmatrix}$ applying $\boldsymbol{R_2 \to R_2 - 3R_1, R_3 \to R_3 - 4R_1}$

$= \begin{vmatrix} 1 & -1 & 0 & x \\ 0 & 1 & 5 & 6-3x \\ 0 & 5 & 7 & 3-4x \\ 0 & 1 & 2 & 3 \end{vmatrix}$ (expanding along 1[st] column)

$= 1\begin{vmatrix} 1 & 5 & 6-3x \\ 5 & 7 & 3-4x \\ 1 & 2 & 3 \end{vmatrix}$ applying $\boldsymbol{R_2 \to R_2 - 5R_1,\ R_3 \to R_3 - R_1}$

$= \begin{vmatrix} 1 & 5 & 6-3x \\ 0 & -18 & 11x-27 \\ 0 & -3 & 3x-3 \end{vmatrix}$ (expanding along 1[st] column)

$$|A| = 1\begin{vmatrix} -18 & 11x-27 \\ -3 & 3x-3 \end{vmatrix} = -18(3x-3) + 3(11x-27)$$

$$\therefore -69 = -21x - 27 \Rightarrow x = 2$$

P1.

Evaluate

$$(i)\ \begin{vmatrix} 1^2 & 2^2 & 3^2 \\ 2^2 & 3^2 & 4^2 \\ 3^2 & 4^2 & 5^2 \end{vmatrix} \qquad (ii)\ \begin{vmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 3^2 & 4^2 & 5^2 & 6^2 \\ 4^2 & 5^2 & 6^2 & 7^2 \end{vmatrix}$$

Solution:

$$(i)\ \begin{vmatrix} 1^2 & 2^2 & 3^2 \\ 2^2 & 3^2 & 4^2 \\ 3^2 & 4^2 & 5^2 \end{vmatrix} = \begin{vmatrix} 1 & 4 & 9 \\ 4 & 9 & 16 \\ 9 & 16 & 25 \end{vmatrix} \ \textit{applying}\ R_2 \to R_2 - 4R_1,\ R_5 \to R_3 - 9R_1$$

$$= \begin{vmatrix} 1 & 4 & 9 \\ 0 & -7 & -20 \\ 0 & -20 & -56 \end{vmatrix}$$

$$= 1\left[(-7)(-56) - (-20)(-20)\right] = 392 - 400 = -8$$

$$(ii)\ \begin{vmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 3^2 & 4^2 & 5^2 & 6^2 \\ 4^2 & 5^2 & 6^2 & 7^2 \end{vmatrix} = \begin{vmatrix} 1 & 4 & 9 & 16 \\ 4 & 9 & 16 & 25 \\ 9 & 16 & 25 & 36 \\ 16 & 25 & 36 & 49 \end{vmatrix} \ \textit{applying}\ R_4 \to R_4 - R_3$$

$$= \begin{vmatrix} 1 & 4 & 9 & 16 \\ 4 & 9 & 16 & 25 \\ 9 & 16 & 25 & 36 \\ 7 & 9 & 11 & 13 \end{vmatrix} \ \textit{applying}\ R_3 \to R_3 - R_2$$

$$= \begin{vmatrix} 1 & 4 & 9 & 16 \\ 4 & 9 & 16 & 25 \\ 5 & 7 & 9 & 11 \\ 7 & 9 & 11 & 13 \end{vmatrix} \ \textit{applying}\ R_2 \to R_2 - R_1$$

$$= \begin{vmatrix} 1 & 4 & 9 & 16 \\ 4 & 9 & 16 & 25 \\ 5 & 7 & 9 & 11 \\ 7 & 9 & 11 & 13 \end{vmatrix} \ \textit{applying}\ R_2 \to R_2 - R_1$$

$$= \begin{vmatrix} 1 & 4 & 9 & 16 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 11 \\ 7 & 9 & 11 & 13 \end{vmatrix} \ \textit{applying}\ R_4 \to R_4 - R_2,\ R_3 \to R_3 - R_2$$

$$= \begin{vmatrix} 1 & 4 & 9 & 16 \\ 3 & 5 & 7 & 9 \\ 2 & 2 & 2 & 2 \\ 4 & 4 & 4 & 4 \end{vmatrix}$$

$$= 0$$

(Since the elements of 4^{th} row and the corresponding elements of 3^{rd} row are in the same ratio)

P2:

If $\boldsymbol{a_1, a_2, a_3, \dots, a_n, \dots}$ are in Geometric Progression, then find the value of the determinant of the matrix $\boldsymbol{A} = \begin{bmatrix} \boldsymbol{lna_n} & \boldsymbol{lna_{n+1}} & \boldsymbol{lna_{n+2}} \\ \boldsymbol{lna_{n+3}} & \boldsymbol{lna_{n+4}} & \boldsymbol{lna_{n+5}} \\ \boldsymbol{lna_{n+6}} & \boldsymbol{lna_{n+7}} & \boldsymbol{lna_{n+8}} \end{bmatrix}$

Solution:

Given $a_1, a_2, a_3, \dots, a_n, \dots$ are in Geometric Progression (G.P)

Let r be the common ratio of the given G.P. Then

$$a_n = ar^{n-1}, a_{n+1} = ar^{n}, a_{n+2} = ar^{n+1}, \dots, \dots$$

Now, $|A| = \begin{vmatrix} lna_n & lna_{n+1} & lna_{n+2} \\ lna_{n+3} & lna_{n+4} & lna_{n+5} \\ lna_{n+6} & lna_{n+7} & lna_{n+8} \end{vmatrix}$

$= \begin{vmatrix} lnar^{n-1} & lnar^{n} & lnar^{n+1} \\ lnar^{n+2} & lnar^{n+3} & lnar^{n+4} \\ lnar^{n+5} & lnar^{n+6} & lnar^{n+7} \end{vmatrix}$ applying $\boldsymbol{C_3 \longrightarrow C_3 - C_2}$

$= \begin{vmatrix} lnar^{n-1} & lnar^{n} & lnr \\ lnar^{n+2} & lnar^{n+3} & lnr \\ lnar^{n+5} & lnar^{n+6} & lnr \end{vmatrix}$ applying $\boldsymbol{C_2 \longrightarrow C_2 - C_1}$

$= \begin{vmatrix} lnar^{n-1} & lnr & lnr \\ lnar^{n+2} & lnr & lnr \\ lnar^{n+5} & lnr & lnr \end{vmatrix}$

$= 0$ (Since the columns $\boldsymbol{C_2, C_3}$ are identical)

$\Longrightarrow |A| = 0$

P3:

Show that $\begin{vmatrix} \mathbf{1} & \boldsymbol{a} & \boldsymbol{a^2} \\ \mathbf{1} & \boldsymbol{b} & \boldsymbol{b^2} \\ \mathbf{1} & \boldsymbol{c} & \boldsymbol{c^2} \end{vmatrix} = (\boldsymbol{a}-\boldsymbol{b})(\boldsymbol{b}-\boldsymbol{c})(\boldsymbol{c}-\boldsymbol{a})$

Solution:

$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$ applying $\boldsymbol{R_2} \to (\boldsymbol{R_2}-\boldsymbol{R_1})$; $\boldsymbol{R_3} \to (\boldsymbol{R_3}-\boldsymbol{R_1})$

$$= \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix}$$

$$= 1.\begin{vmatrix} b-a & b^2-a^2 \\ c-a & c^2-a^2 \end{vmatrix} = (b-a)(c-a)\begin{vmatrix} 1 & b-a \\ 1 & c-a \end{vmatrix}$$

$$= (b-a)(c-a)(c-a-b-a) = (b-a)(c-a)(c-b)$$

$$= (a-b)(b-c)(c-a)$$

$$\therefore \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$$

Note 1:

$\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix} = xyz\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$ (why?)

$= xyz\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$ (why?)

$$= xyz(x-y)(y-z)(z-x)$$

Note 2: If x, y, z are pairwise distinct and $\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & y^3 & 1+z^3 \end{vmatrix} = 0$, then prove that $xyz = -1$

P4:

If $\begin{vmatrix} \boldsymbol{x-2} & \boldsymbol{2x-3} & \boldsymbol{3x-4} \\ \boldsymbol{x-4} & \boldsymbol{2x-9} & \boldsymbol{3x-16} \\ \boldsymbol{x-8} & \boldsymbol{2x-27} & \boldsymbol{3x-64} \end{vmatrix} = \boldsymbol{0}$ **then find the value of** $\boldsymbol{x}$**.**

Solution:

$$0 = \begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} \quad \text{applying } \boldsymbol{R_2 \to (R_2 - R_1), R_3 \to (R_3 - R_1)}$$

$$= \begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ -2 & -6 & -12 \\ -6 & -24 & -60 \end{vmatrix}$$

$$= (-2)(-6)\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ 1 & 3 & 6 \\ 1 & 4 & 10 \end{vmatrix} \qquad \text{applying } \boldsymbol{R_2 \to (R_2 - R_3)}$$

$$= (-2)(-6)\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ 0 & -1 & -4 \\ 1 & 4 & 10 \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ 0 & -1 & -4 \\ 1 & 4 & 10 \end{vmatrix} = 0$$

$$\Rightarrow (x-2)(-10+16) + (1)\{(2x-3)(-4) + (3x-4)\} = 0$$

(expanding along 1st column)

$$\Rightarrow 6(x-2) - 4(2x-3) + (3x-4) = 0$$

$$\Rightarrow x - 4 = 0 \Rightarrow x = 4.$$

IP1.

For a fixed positive integer n, if $D = \begin{vmatrix} (n-1)! & (n+1)! & \frac{(n+3)!}{n(n+1)} \\ (n+1)! & (n+3)! & \frac{(n+5)!}{(n+2)(n+3)} \\ (n+3)! & (n+5)! & \frac{(n+7)!}{(n+4)(n+5)} \end{vmatrix}$, then find the value of $\frac{D!}{(n-1)!(n+1)!(n+3)!}$

Solution:

$$D = \begin{vmatrix} (n-1)! & (n+1)! & \frac{(n+3)!}{n(n+1)} \\ (n+1)! & (n+3)! & \frac{(n+5)!}{(n+2)(n+3)} \\ (n+3)! & (n+5)! & \frac{(n+7)!}{(n+4)(n+5)} \end{vmatrix}$$

$$D = (n-1)!(n+1)!(n+3)! \begin{vmatrix} 1 & (n+1)n & (n+3)(n+2) \\ 1 & (n+3)(n+2) & (n+5)(n+4) \\ 1 & (n+5)(n+4) & (n+7)(n+6) \end{vmatrix}$$

$$\frac{D}{(n-1)!(n+1)!(n+3)!} = \begin{vmatrix} 1 & (n+1)n & (n+3)(n+2) \\ 1 & (n+3)(n+2) & (n+5)(n+4) \\ 1 & (n+5)(n+4) & (n+7)(n+6) \end{vmatrix}$$

applying $R_3 \longrightarrow R_3 - R_1$ and $R_2 \longrightarrow R_2 - R_1$

$$= \begin{vmatrix} 1 & (n+1)n & (n+3)(n+2) \\ 0 & 4n+6 & 4n+14 \\ 0 & 8n+20 & 8n+36 \end{vmatrix}$$ applying $R_3 \longrightarrow R_3 - 2R_2$

$$= \begin{vmatrix} 1 & (n+1)n & (n+3)(n+2) \\ 0 & 4n+6 & 4n+14 \\ 0 & 8 & 8 \end{vmatrix}$$ (expanding along 1st column)

$$= 1[32n + 48 - 32n - 112] = -64$$

$$\Rightarrow \frac{D}{(n-1)!\,(n+1)!\,(n+3)!} = -64$$

IP2:

For all values of $\boldsymbol{A, B, C}$ and $\boldsymbol{P, Q, R}$, show that

$$\begin{vmatrix} \cos(A-P) & \cos(A-Q) & \cos(A-R) \\ \cos(B-P) & \cos(B-Q) & \cos(B-R) \\ \cos(C-P) & \cos(C-Q) & \cos(C-R) \end{vmatrix} = 0$$

Solution:

$$\begin{vmatrix} \cos(A-P) & \cos(A-Q) & \cos(A-R) \\ \cos(B-P) & \cos(B-Q) & \cos(B-R) \\ \cos(C-P) & \cos(C-Q) & \cos(C-R) \end{vmatrix}$$

$$= \begin{vmatrix} cosAcosP + sinAsinP & \cos(A-Q) & \cos(A-R) \\ cosBcosP + sinBsinP & \cos(B-Q) & \cos(B-R) \\ cosCsinP + sinCsinP & \cos(C-Q) & \cos(C-R) \end{vmatrix}$$

$$= cosP \begin{vmatrix} cosA & \cos(A-Q) & \cos(A-R) \\ cosB & \cos(B-Q) & \cos(B-R) \\ cosC & \cos(C-Q) & \cos(C-R) \end{vmatrix} + sinP \begin{vmatrix} sinA & \cos(A-Q) & \cos(A-R) \\ sinB & \cos(B-Q) & \cos(B-R) \\ sinC & \cos(C-Q) & \cos(C-R) \end{vmatrix}$$

applying $\boldsymbol{C_2 \to C_2 - C_1 cosQ}$; $\boldsymbol{C_3 \to C_3 - C_1 cosR}$ on first determinant and $\boldsymbol{C_2 \to C_2 - C_1 sinQ}$; $\boldsymbol{C_3 \to C_3 - C_1 sinR}$ on second determinant, we get

$$= cosP \begin{vmatrix} cosA & sinAsinQ & sinAsinR \\ cosB & sinBsinQ & sinBsinR \\ cosC & sinCsinQ & sinBsinR \end{vmatrix} + sinP \begin{vmatrix} sinA & cosAcosQ & cosAcosR \\ sinB & cosBcosQ & cosBcosR \\ sinC & cosCcosQ & cosCcosR \end{vmatrix}$$

$$= cosPsinQsinR \begin{vmatrix} cosA & sinA & sinA \\ cosB & sinB & sinB \\ cosC & sinC & sinB \end{vmatrix} + sinPcosQcosR \begin{vmatrix} sinA & cosA & cosA \\ sinB & cosB & cosB \\ sinC & cosC & cosC \end{vmatrix}$$

$= 0 + 0 = 0$ (Since the columns $\boldsymbol{C_2, C_3}$ are same in both determinants)

Aliter:

For all values of $\boldsymbol{A, B, C}$ and $\boldsymbol{P, Q, R}$, show that

$$\begin{vmatrix} \cos(A-P) & \cos(A-Q) & \cos(A-R) \\ \cos(B-P) & \cos(B-Q) & \cos(B-R) \\ \cos(C-P) & \cos(C-Q) & \cos(C-R) \end{vmatrix} = 0$$

Solution:

Note that $A = \begin{bmatrix} \cos(A-P) & \cos(A-Q) & \cos(A-R) \\ \cos(B-P) & \cos(B-Q) & \cos(B-R) \\ \cos(C-P) & \cos(C-Q) & \cos(C-R) \end{bmatrix}$

$$= \begin{bmatrix} cosA & sinA & 0 \\ cosB & sinB & 0 \\ cosC & sinC & 0 \end{bmatrix} + \begin{bmatrix} cosP & cosQ & cosR \\ sinP & sinQ & sinR \\ 1 & 1 & 1 \end{bmatrix}$$

$$= BC$$

Now, $detA = detBC = detB.detC$

$= 0 \quad (\because detB = 0)$

Hence, $detA = 0$

IP3:

Show that $\begin{vmatrix} \mathbf{1} & \boldsymbol{a^2} & \boldsymbol{a^3} \\ \mathbf{1} & \boldsymbol{b^2} & \boldsymbol{b^3} \\ \mathbf{1} & \boldsymbol{c^2} & \boldsymbol{c^3} \end{vmatrix} = (\boldsymbol{a}-\boldsymbol{b})(\boldsymbol{b}-\boldsymbol{c})(\boldsymbol{c}-\boldsymbol{a})(\boldsymbol{ab}+\boldsymbol{bc}+\boldsymbol{ca})$

Solution:

$\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$ applying $\boldsymbol{R_1 \to R_1 - R_3}$, $\boldsymbol{R_2 \to R_2 - R_3}$

$$= \begin{vmatrix} 0 & a^2-c^2 & a^3-c^3 \\ 0 & b^2-c^2 & b^3-c^3 \\ 1 & c^2 & c^3 \end{vmatrix}$$

$$= (a-c)(b-c)\begin{vmatrix} 0 & a+c & a^2+ac+c^2 \\ 0 & b+c & b^2+bc+c^2 \\ 1 & c^2 & c^3 \end{vmatrix}$$ applying $\boldsymbol{R_2 \to R_2 - R_1}$

$$= (a-c)(b-c)\begin{vmatrix} 0 & a+c & a^2+ac+c^2 \\ 0 & b-a & b^2-a^2+bc-ac \\ 1 & c^2 & c^3 \end{vmatrix}$$

$$= (a-c)(b-c)(b-a)\begin{vmatrix} 0 & a+c & a^2+ac+c^2 \\ 0 & 1 & b+a+c \\ 1 & c^2 & c^3 \end{vmatrix}$$

$$= (a-c)(b-c)(b-a)\begin{vmatrix} a+c & a^2+ac+c^2 \\ 1 & a+b+c \end{vmatrix}$$

$$= (a-b)(b-c)(c-a)[(a+c)(a+b+c)-(a^2+ac+c^2)]$$

$$= (a-b)(b-c)(c-a)(ab+bc+ca)$$

$$\therefore \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$$

Note:

If $\boldsymbol{A} = \begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{c} \\ \boldsymbol{a^2} & \boldsymbol{b^2} & \boldsymbol{c^2} \\ \boldsymbol{bc} & \boldsymbol{ca} & \boldsymbol{ab} \end{bmatrix}$ and $\boldsymbol{B} = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \boldsymbol{a^2} & \boldsymbol{b^2} & \boldsymbol{c^2} \\ \boldsymbol{a^3} & \boldsymbol{b^3} & \boldsymbol{c^3} \end{bmatrix}$, then show that

$$|\boldsymbol{A}| = |\boldsymbol{B}| = (\boldsymbol{a}-\boldsymbol{b})(\boldsymbol{b}-\boldsymbol{c})(\boldsymbol{c}-\boldsymbol{a})(\boldsymbol{ab}+\boldsymbol{bc}+\boldsymbol{ca})$$

Solution:

We have $|A| = \begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix}$

Multiplying $\boldsymbol{C_1}, \boldsymbol{C_2}$ and $\boldsymbol{C_3}$ by a, b and c respectively, we get

$$|A| = \frac{1}{abc}\begin{vmatrix} a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \\ abc & abc & abc \end{vmatrix} = \frac{abc}{abc}\begin{vmatrix} a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = |B|$$

Note that $|B| = |B^T|$ i.e., $|B| = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$

$$\therefore |A| = |B| = (a-b)(b-c)(c-a)(ab+bc+ca)$$

IP4.

Show that $\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{vmatrix} = abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$

Solution:

$$\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{vmatrix}$$

Multiplying the elements of C_1, C_2, C_3 and C_4 by $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ and $\frac{1}{d}$ respectively and multiplying the determinant by $abcd$ to compensate it

$$= abcd\begin{vmatrix} \frac{1}{a}+1 & \frac{1}{b} & \frac{1}{c} & \frac{1}{d} \\ \frac{1}{a} & \frac{1}{b}+1 & \frac{1}{c} & \frac{1}{d} \\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c}+1 & \frac{1}{d} \\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c} & \frac{1}{d}+1 \end{vmatrix}$$

applying $C_1 \to C_1 + C_2 + C_3 + C_4$

$$= abcd\begin{vmatrix} 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d} & \frac{1}{b} & \frac{1}{c} & \frac{1}{d} \\ 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d} & \frac{1}{b}+1 & \frac{1}{c} & \frac{1}{d} \\ 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d} & \frac{1}{b} & \frac{1}{c}+1 & \frac{1}{d} \\ 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d} & \frac{1}{b} & \frac{1}{c} & \frac{1}{d}+1 \end{vmatrix}$$

$$= abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)\begin{vmatrix} 1 & \frac{1}{b} & \frac{1}{c} & \frac{1}{d} \\ 1 & \frac{1}{b}+1 & \frac{1}{c} & \frac{1}{d} \\ 1 & \frac{1}{b} & \frac{1}{c}+1 & \frac{1}{d} \\ 1 & \frac{1}{b} & \frac{1}{c} & \frac{1}{d}+1 \end{vmatrix}$$

applying $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1, R_4 \to R_4 - R_1$

$$= abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)\begin{vmatrix} 1 & \frac{1}{b} & \frac{1}{c} & \frac{1}{d} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$$

(The *det* is the product of diagonal entries, as it is the *det* of an upper triangular matrix).

10.5. Evaluation of Determinants using Properties

Exercises

1. Show that $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} = 0.$

2. Show that $\begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix} = 0$

3. Without expanding prove that $\begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix} = 0.$

4. Show that $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$

5. Without expanding evaluate the determinant $\begin{vmatrix} \sin\alpha & \cos\alpha & \sin(\alpha+\delta) \\ \sin\beta & \cos\beta & \sin(\beta+\delta) \\ \sin\gamma & \cos\gamma & \sin(\gamma+\delta) \end{vmatrix}$

6. Find the determinant of the matrix $A = \begin{bmatrix} 2 & -1 & 3 & 0 \\ 5 & 4 & -2 & 1 \\ 3 & 1 & 0 & 2 \\ 4 & -5 & 6 & -1 \end{bmatrix}$

7. Show that $\begin{vmatrix} (b+c)^2 & ba & ca \\ ab & (c+a)^2 & cb \\ ac & bc & (a+b)^2 \end{vmatrix} = 2abc(a+b+c)^3$

8. Show that $\begin{vmatrix} a & b-c & c+b \\ a+c & b & c-a \\ a-b & b+a & c \end{vmatrix} = (a+b+c)(a^2+b^2+c^2)$

9. Show that $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$

10. If ω is complex (non-real) cube root of 1, then show that $\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix} = 0$

11. If x, y, z are different from zero and $\begin{vmatrix} a & b-y & c-z \\ a-x & b & c-z \\ a-x & b-y & c \end{vmatrix} = 0$, then find the value of the expression $\frac{a}{x} + \frac{b}{y} + \frac{c}{z}$

12. Let $\omega = \frac{-1+i\sqrt{3}}{2}$. Then the value of the determinant $\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1-\omega^2 & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{vmatrix}$ is

A. 3ω B. $3\omega(\omega-1)$ C. $3\omega^2$ D. $3\omega(\omega-1)$: Ans. B

13. For positive numbers x, y and z, find the numerical value of the determinant

$$\begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix}$$

14. Find the value of the determinant $\begin{vmatrix} xp+y & x & y \\ yp+z & y & z \\ 0 & xp+y & yp+z \end{vmatrix}$.

15. If $\begin{vmatrix} 6i & -3i & 1 \\ 4 & 3i & -1 \\ 20 & 3 & i \end{vmatrix} = x+iy$, then $(x,y) =$

A. $(3,1)$ B. $(1,3)$ C. $(0,3)$ D. $(0,0)$; Ans. D

5.8 Finding Inverse of a Matrix

Learning objectives:

- ☆ To define the adjoint of a square matrix.
- ☆ To derive a formula for the inverse of a non-singular matrix in terms of its adjoint.

  AND
- ☆ To practice the related problems.

We develop a formula for the inverse of a nonsingular matrix.

Definition

Let $A = (a_{ij})_{n\times n}$ be an $n \times n$ matrix and C_{ij} be the cofactor of a_{ij}. The matrix whose (i,j)th element is C_{ij} is called the **matrix of cofactors** of A. The transpose of this matrix is called the ***adjoint of A*** and is denoted $adj(A)$.

$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix} \qquad adj(A) = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}'$$

matrix of cofactors — adjoint matrix

Example

Give the matrix of cofactors and the adjoint matrix of the following matrix.

$$A = \begin{bmatrix} 2 & 0 & 3 \\ -1 & 4 & -2 \\ 1 & -3 & 5 \end{bmatrix}$$

Solution

The cofactors of A are as follows.

$$C_{11} = \begin{vmatrix} 4 & -2 \\ -3 & 5 \end{vmatrix} = 14, \quad C_{12} = -\begin{vmatrix} -1 & -2 \\ 1 & 5 \end{vmatrix} = 3, \quad C_{12} = \begin{vmatrix} -1 & 4 \\ 1 & -3 \end{vmatrix} = -1$$

Similarly,

$$C_{21} = -9, \qquad C_{22} = 7, \qquad C_{23} = 6$$
$$C_{31} = -12, \qquad C_{32} = 1, \qquad C_{33} = 8$$

The matrix of cofactors of A is

$$\begin{bmatrix} 14 & 3 & -1 \\ -9 & 7 & 6 \\ -12 & 1 & 8 \end{bmatrix}$$

The adjoint of A is the transpose of this matrix.

$$\text{adj}(A) = \begin{bmatrix} 14 & -9 & -12 \\ 3 & 7 & 1 \\ -1 & 6 & 8 \end{bmatrix}$$

Theorem:

If A is a square matrix of order n then

$$A.\,adj(A) = adj(A).\,A = |A|I_n$$

Proof:

Let $A = (a_{ij})_{n\times n}$. We have

$$adj(A) = transpose\ of \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$$

$$adj(A) = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

Now, consider $A.\,adj(A)$. The $(i,j)^{th}$ element of this product is:

$(i,j)^{th} element = (i^{th} row\ of\ A) \times \left(j^{th} column\ of\ adj(A)\right)$

$$= \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix} \begin{bmatrix} C_{j1} \\ C_{j2} \\ \vdots \\ C_{jn} \end{bmatrix}$$

$$= a_{i1}C_{j1} + a_{i2}C_{j2} + \cdots + a_{in}C_{jn}$$

$$= \begin{cases} |A|, & if\ i = j \\ 0, & if\ i \neq j \end{cases}$$

The product of $A.\,adj(A)$ is thus a diagonal matrix with the diagonal elements all being $|A|$.

$$\therefore A.\,adj(A) = |A|I_n$$

Similarly, $adj(A).\,A = |A|I_n$

Thus, $A.\,adj(A) = adj(A).\,A = |A|I_n$

Theorem:

The inverse of a square matrix A exists if and only if A is non-singular.

(A square matrix A is invertible if and only if $|A| \neq 0$)

Proof:

Let A be a square matrix of order n.Suppose the inverse of A exists, say B

Therefore, $AB = BA = I_n$

$$\Rightarrow |AB| = |I_n| \Rightarrow |A||B| = 1$$
$$\Rightarrow |A| \neq 0 \Rightarrow A \text{ is non-singular.}$$

Conversely, suppose that A is non-singular.Then $|A| \neq 0$. We have

$$A.\,adj(A) = adj(A).\,A = |A|I_n$$

$$\Rightarrow A\left(\frac{1}{|A|}adj(A)\right) = \left(\frac{1}{|A|}adj(A)\right)A = I_n \quad (\text{Since } |A| \neq 0)$$

$\Rightarrow \left(\frac{1}{|A|}adj(A)\right)$ is the inverse of A. Thus, the inverse of A exists.

Hence the theorem.

Note:

1. If A is non-singular then $\boldsymbol{A^{-1} = \frac{1}{|A|}adj(A)}$
2. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is non-singular then $\boldsymbol{A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$

   Proof:

   Given $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is non-singular

   $$\Rightarrow |A| = ad - bc \neq 0$$

   Therefore, A^{-1} exists.

   Now, $C_{11} = (-1)^{1+1}M_{11} = d,\ C_{12} = (-1)^{1+2}M_{12} = -c$

   $$C_{21} = (-1)^{2+1}M_{21} = -b,\ C_{22} = (-1)^{2+2}M_{22} = a$$

   $$\therefore\ adj(A) = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}' = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}' = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

   $$\therefore\ A^{-1} = \frac{1}{|A|}adj(A) = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

   Thus, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, if $ad - bc \neq 0$

Example

Find out which of the following matrices are invertible.

$$A = \begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix},\ B = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix},\ C = \begin{bmatrix} 2 & 4 & -3 \\ 4 & 12 & -7 \\ -1 & 0 & 1 \end{bmatrix},\ D = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & 2 \\ 2 & 8 & 0 \end{bmatrix}$$

Solution

$|A| = 5 \neq 0 \Rightarrow A$ is non-singular. Therefore, A is invertible

$|B| = 0 \Rightarrow B$ is singular. Therefore, inverse does not exist.

$|C| = 0 \Rightarrow C$ is singular. Therefore, inverse does not exist.

$|D| = 2 \neq 0 \Rightarrow D$ is non-singular. Therefore, D is invertible.

Example:

Compute the inverse of the matrix

$$A = \begin{bmatrix} 2 & 0 & 3 \\ -1 & 4 & -2 \\ 1 & -3 & 5 \end{bmatrix}$$

Solution

$|A| = 25 \neq 0$. Thus the inverse of A exists.

$$\text{adj}(A) = \begin{bmatrix} 14 & -9 & 12 \\ 3 & 7 & 1 \\ -1 & 6 & 8 \end{bmatrix}$$

Therefore,

$$A^{-1} = \frac{1}{|A|}\text{adj}(A) = \frac{1}{25}\text{adj}(A)$$

$$= \begin{bmatrix} \frac{14}{25} & -\frac{9}{25} & -\frac{12}{25} \\ \frac{3}{25} & \frac{7}{25} & \frac{1}{25} \\ -\frac{1}{25} & \frac{6}{25} & \frac{8}{25} \end{bmatrix}$$

Example:

Compute the inverse of the matrix $A = \begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix}$

Solution:

$$|A| = \begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix} = 2 + 3 = 5 \neq 0$$

$\Rightarrow A$ is non-singular. Therefore, A has inverse and

$$\therefore A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 2 & 1 \\ -3 & 1 \end{pmatrix}$$

IP1.

If $\boldsymbol{A} = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{5} \\ \mathbf{2} & \mathbf{3} & \mathbf{1} \\ -\mathbf{1} & \mathbf{1} & \mathbf{1} \end{bmatrix}$ **then find** $\boldsymbol{adjA'}$

Solution:

Step1:

Given $A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix}$

Then we have $A' = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & 1 \\ 5 & 1 & 1 \end{bmatrix}$

We have,

$adjA' =$ Transpose of the cofactor matrix C of A'

Now, we find the cofactor matrix C of A'

$$C_{11} = (-1)^{1+1} \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} = [3 - 1] = 2$$

$$C_{12} = (-1)^{1+2} \begin{vmatrix} 2 & 1 \\ 5 & 1 \end{vmatrix} = -[2 - 5] = 3$$

$$C_{31} = (-1)^{1+3} \begin{vmatrix} 2 & 3 \\ 5 & 1 \end{vmatrix} = [2 - 15] = -13$$

$$C_{21} = (-1)^{2+1} \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} = -[2 + 1] = -3$$

$$C_{22} = (-1)^{2+2} \begin{vmatrix} 1 & -1 \\ 5 & 1 \end{vmatrix} = [1 + 5] = 6$$

$$C_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 2 \\ 5 & 1 \end{vmatrix} = -[1 - 10] = 9$$

$$C_{3+1} = (-1)^{3+1} \begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} = [2 + 3] = 5$$

$$C_{32} = (-1)^{3+2} \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = -[1 + 2] = -3$$

$$C_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = [3 - 4] = -1$$

$$C = \begin{bmatrix} 2 & 3 & -13 \\ -3 & 6 & 9 \\ 5 & -3 & -1 \end{bmatrix}$$

$$adjA' = C' = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 6 & -3 \\ -13 & 9 & -1 \end{bmatrix}$$

IP2.

Find the inverse of $A = \begin{bmatrix} \mathbf{1} & \mathbf{3} & \mathbf{3} \\ \mathbf{1} & \mathbf{4} & \mathbf{3} \\ \mathbf{1} & \mathbf{3} & \mathbf{4} \end{bmatrix}$

Solution:

Step1:

$$|A| = \begin{vmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{vmatrix}$$

$$= (16 - 9) - 3(4 - 3) + 3(3 - 4) = 1 \neq 0.$$

Therefore, A is invertible.

Step2:

Let C_{ij} be the cofactor of a_{ij} in $A = [a_{ij}]_{3\times3}$, then

$$C_{11} = (-1)^{1+1} \begin{vmatrix} 4 & 3 \\ 3 & 4 \end{vmatrix} = 7$$

$$C_{12} = (-1)^{1+2} \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = -1$$

$$C_{13} = (-1)^{1+3} \begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} = -1$$

$$C_{21} = (-1)^{2+1} \begin{vmatrix} 3 & 3 \\ 3 & 4 \end{vmatrix} = -3$$

$$C_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = 1$$

$$C_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} = 0$$

$$C_{31} = (-1)^{3+1} \begin{vmatrix} 3 & 3 \\ 4 & 3 \end{vmatrix} = -3$$

$$C_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} = 0$$

$$C_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = 1$$

$$C = \text{Cofactor matrix of } A = \begin{bmatrix} 7 & -1 & -1 \\ -3 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

Step3:

$$\therefore adjA = C' = \begin{bmatrix} 7 & -1 & -1 \\ -3 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}' = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Step4:

We have $A^{-1} = \frac{1}{|A|}\, adjA$

$$A^{-1} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

IP3.

Find the matrix A satisfying the matrix equation

$$\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} A \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Solution:

Step1:

Given $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} A \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Let $B = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix}$

$|B| = \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = 4 - 3 = 1 \neq 0$ and

$|C| = \begin{vmatrix} -3 & 2 \\ 5 & -3 \end{vmatrix} = 9 - 10 = -1 \neq 0$

$\Longrightarrow B$ and C are non-singular matrices.

$\therefore$ B^{-1} and C^{-1} exists.

Step2:

Now, $BAC = I$

$$\Longrightarrow B^{-1}(BAC)C^{-1} = B^{-1}IC^{-1}$$

$$\Longrightarrow (B^{-1}B)A(CC^{-1}) = B^{-1}C^{-1}$$

$$\Longrightarrow |A| = B^{-1}C^{-1} \Longrightarrow A = B^{-1}C^{-1}$$

Step3:

$$B^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$$

$$C^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = -\begin{bmatrix} -3 & -2 \\ -5 & -3 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}$$

$$\therefore A = B^{-1}C^{-1} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

IP4.

Let $\boldsymbol{A}$ be a square matrix order $\boldsymbol{n}$.

i. If $|\boldsymbol{A}| = \boldsymbol{0}$ then $\boldsymbol{A}.\,\boldsymbol{adjA} = \boldsymbol{adjA}.\,\boldsymbol{A} = \boldsymbol{0}$

ii. If $|\boldsymbol{A}| \neq \boldsymbol{0}$ then $|\boldsymbol{adj}(\boldsymbol{A})| = |\boldsymbol{A}|^{\boldsymbol{n-1}}$

iii. If $|\boldsymbol{A}| \neq \boldsymbol{0}$ then $|\boldsymbol{adj}(\boldsymbol{adjA})| = |\boldsymbol{A}|^{(\boldsymbol{n-1})^2}$

Solution:

If A is a square matrix of order n then

$$A.\,adjA = adjA.\,A = |A|I_n \text{ .... } (1)$$

i. Let $|A| = 0$ then from (1)

$$A.\,adjA = adjA.\,A = 0$$

ii. Let $|A| \neq 0$ then

$(1) \Longrightarrow |A.\,adjA| = \big||A|I_n\big|$

$\Longrightarrow |A||adjA| = |A|^n$

$\Longrightarrow |adjA| = |A|^{n-1} \qquad (\because\ |A| \neq 0)$

iii. Let $|A| \neq 0$. We have

$(adjA)\big(adj(adjA)\big) = \big(adj(adjA)\big).\,(adjA) = |adjA|I_n$

Taking determinants on both sides, we get

$$\big|(adjA)\big(adj(adjA)\big)\big| = \big||adjA|.\,I_n\big|$$

$\Longrightarrow |adjA||adj(adjA)| = |adjA|^n$

$\Longrightarrow |A|^{n-1}|adj(adjA)| = (|A|^{n-1})^n \quad (\because |adjA| = |A|^{n-1})$

$\Longrightarrow |adj(adjA)| = |A|^{n(n-1)}|A|^{-(n-1)} = |A|^{(n-1)^2}$

$\therefore\ |adj(adjA)| = |A|^{(n-1)^2}$, if $|A| \neq 0$

P1.

If $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 2 & -1 \\ -4 & 5 & 2 \end{bmatrix}$ then find $adjA =$

Solution:

$$\text{Given } A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 2 & -1 \\ -4 & 5 & 2 \end{bmatrix}$$

We have, $adj(A)$ = Transpose of the cofactor matrix of A

Now, we find the cofactor matrix C of A

$$C_{11} = (-1)^{1+1}\begin{vmatrix} 2 & -1 \\ 5 & 2 \end{vmatrix} = [4 + 5] = 9$$

$$C_{12} = (-1)^{1+2}\begin{vmatrix} 0 & -1 \\ -4 & 2 \end{vmatrix} = -[0 - 4] = 4$$

$$C_{13} = (-1)^{1+3}\begin{vmatrix} 0 & 2 \\ -4 & 5 \end{vmatrix} = [0 + 8] = 8$$

$$C_{21} = (-1)^{2+1}\begin{vmatrix} -2 & 3 \\ 5 & 2 \end{vmatrix} = -[-4 - 15] = 19$$

$$C_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 3 \\ -4 & 2 \end{vmatrix} = [2 + 12] = 14$$

$$C_{23} = (-1)^{2+3}\begin{vmatrix} 1 & -2 \\ -4 & 5 \end{vmatrix} = -[5 - 8] = 3$$

$$C_{31} = (-1)^{3+1}\begin{vmatrix} -2 & 3 \\ 2 & -1 \end{vmatrix} = [2 - 6] = -4$$

$$C_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 3 \\ 0 & -1 \end{vmatrix} = -[-1 - 0] = 1$$

$$C_{33} = (-1)^{3+3}\begin{vmatrix} 1 & -2 \\ 0 & 2 \end{vmatrix} = [2 - 0] = 2$$

$$\therefore \; C = \begin{bmatrix} 9 & 4 & 8 \\ 19 & 14 & 3 \\ -4 & 1 & 2 \end{bmatrix}$$

$$\therefore \; adj(A) = C' = \begin{bmatrix} 9 & 19 & -4 \\ 4 & 14 & 1 \\ 8 & 3 & 2 \end{bmatrix}$$

P2.

The inverse of the matrix $A = \begin{bmatrix} 1 & 0 & -2 \\ -2 & -1 & 2 \\ 3 & 4 & 1 \end{bmatrix}$ is _______

Solution:

Given matrix is $A = \begin{bmatrix} 1 & 0 & -2 \\ -2 & -1 & 2 \\ 3 & 4 & 1 \end{bmatrix}$

$$|A| = 1(-1-8) - 2(-8+3) = 1 \neq 0$$

$\Longrightarrow A$ is a non-singular matrix. Therefore, A is invertible.

Let C_{ij} be the cofactor of a_{ij} in $A = (a_{ij})_{3\times 3}$ then

$$C_{11} = (-1)^{1+1}\begin{vmatrix} -1 & 2 \\ 4 & 1 \end{vmatrix} = [-1-8] = -9$$

$$C_{12} = (-1)^{1+2}\begin{vmatrix} -2 & 2 \\ 3 & 1 \end{vmatrix} = -[-2-6] = 8$$

$$C_{13} = (-1)^{1+3}\begin{vmatrix} -2 & -1 \\ 3 & 4 \end{vmatrix} = [-8+3] - 5$$

$$C_{21} = (-1)^{2+1}\begin{vmatrix} 0 & -2 \\ 4 & 1 \end{vmatrix} = -[0+8] = -8$$

$$C_{22} = (-1)^{2+2}\begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} = [1+6] = 7$$

$$C_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 0 \\ 3 & 4 \end{vmatrix} = -[4-0] = -4$$

$$C_{31} = (-1)^{3+1}\begin{vmatrix} 0 & -2 \\ -1 & 2 \end{vmatrix} = [0-2] = -2$$

$$C_{32} = (-1)^{3+2}\begin{vmatrix} 1 & -2 \\ -2 & 2 \end{vmatrix} = -[2-4] = 2$$

$$C_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 0 \\ -2 & -1 \end{vmatrix} = [-1-0] = -1$$

$$\therefore C = cofactor\ matrix\ of\ A = \begin{bmatrix} -9 & 8 & -5 \\ -8 & 7 & -4 \\ -2 & 2 & -1 \end{bmatrix}$$

$$\therefore \ adjA = C' = \begin{bmatrix} -9 & -8 & -2 \\ 8 & 7 & 2 \\ -5 & -4 & -1 \end{bmatrix}$$

$$\therefore The\ inverse\ of\ A\ is\ \ A^{-1} = \frac{1}{|A|}Adj(A)$$

$$\therefore A^{-1} = \begin{bmatrix} -9 & -8 & -2 \\ 8 & 7 & 2 \\ -5 & -4 & -1 \end{bmatrix}$$

P3.

Find the matrix B such that $\begin{bmatrix} 1 & -4 \\ 3 & -2 \end{bmatrix} B = \begin{bmatrix} 16 & -6 \\ 7 & 2 \end{bmatrix}$

Solution:

Given $\begin{bmatrix} 1 & -4 \\ 3 & -2 \end{bmatrix} B = \begin{bmatrix} 16 & -6 \\ 7 & 2 \end{bmatrix}$

Let $A = \begin{bmatrix} 1 & -4 \\ 3 & -2 \end{bmatrix}$ and $C = \begin{bmatrix} 16 & -6 \\ 7 & 2 \end{bmatrix}$

$|A| = \begin{vmatrix} 1 & -4 \\ 3 & -2 \end{vmatrix} = -2 + 12 = 10 \neq 0$

$\Longrightarrow A$ is non-singular. Therefore, A^{-1} exists.

Now, $AB = C \Longrightarrow A^{-1}(AB) = A^{-1}C$

$$\Longrightarrow (A^{-1}A)B = A^{-1}C$$

$$\Longrightarrow IB = A^{-1}C \Rightarrow B = A^{-1}C$$

Now, $A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{10}\begin{bmatrix} -2 & 4 \\ -3 & 1 \end{bmatrix}$

and $B = A^{-1}C = \frac{1}{10}\begin{bmatrix} -2 & 4 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 16 & -6 \\ 7 & 2 \end{bmatrix}$

$$B = \frac{1}{10}\begin{bmatrix} 32 + 28 & 12 + 8 \\ 48 + 7 & 18 + 2 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ \frac{11}{2} & 2 \end{bmatrix}$$

P4.

i. If A is a square matrix such that $A.adj(A) = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

then $\det\left(adj(A)\right) =$

ii. If A is a non-singular $n \times n$ matrix and $adj\left(adj(A)\right) = kA$

then $k =$

Solution:

i. We have $A.adj(A) = 4\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} = 4I_3$

$\Longrightarrow A$ is a 3×3 matrix and $|A| = 4$

$\therefore A$ is a non-singular matrix of order 3.

$\therefore \det\big(adj(A)\big) = |A|^{n-1} = 4^{3-1} = 16$

ii. Given A is a non-singular matrix of order n and $adj\big(adj(A)\big) = kA$.

We have $|adj(adjA)| = |A|^{(n-1)^2}$

Now, $adj(adjA) = kA \Longrightarrow |adj(adjA)| = |kA|$

$$\Longrightarrow |A|^{(n-1)^2} = k^n|A|$$

$$\Longrightarrow k^n = |A|^{(n-1)^2-1} = |A|^{n(n-2)}$$

$$\Longrightarrow k = |A|^{(n-2)}$$

1. Verify that $(adjA)A = |A|I = A(adjA)$ for the following matrices:

$(i)\ \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ $(ii)\ \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix}$ $(iii)\ \begin{bmatrix} 2 & -1 & 3 \\ 4 & 2 & 5 \\ 0 & 4 & -1 \end{bmatrix}$

$(iv)\ \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 1 & 1 & 3 \end{bmatrix}$ $(v)\ \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 2 & 4 & 3 \end{bmatrix}$

2. If $A = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$, show that $adj\ A = 3A'$

3. Find the inverse of each of the following matrices:

(i) $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(iii) $\begin{bmatrix} a & b \\ c & \frac{1+bc}{a} \end{bmatrix}$ (iv) $\begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix}$

(v) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$ (vi) $\begin{bmatrix} 1 & 2 & 5 \\ 1 & -1 & -1 \\ 2 & 3 & -1 \end{bmatrix}$

(vii) $\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ (viii) $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

(ix) $\begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$ (x) $\begin{bmatrix} 0 & 0 & -1 \\ 3 & 4 & 5 \\ -2 & -4 & -7 \end{bmatrix}$

(xi) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{bmatrix}$

4. Find the matrix A satisfying the matrix equation

a. $\begin{bmatrix} 5 & 4 \\ 1 & 1 \end{bmatrix} A = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$

b. $\begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix} A \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 0 & 4 \end{bmatrix}$

c. $A \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 14 & 7 \\ 7 & 7 \end{bmatrix}$

d. $\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

10.9. Rank of a Matrix

Learning objectives:

- To define the rank of the matrix by elementary transformation
  And
- To solve the related problems

r-rowed minor of a matrix

Let A be a given matrix. If B is a square submatrix of order r of A, then $\det B$ is called ***an r-rowed minor of A***.

Example: If $A = \begin{bmatrix} 0 & 1 & 1 & -2 \\ 4 & 0 & 2 & 5 \\ 2 & 1 & 3 & 1 \end{bmatrix}$, then $\begin{vmatrix} 0 & 1 & -2 \\ 4 & 0 & 5 \\ 2 & 3 & 1 \end{vmatrix}$ is 3 – rowed minor of A and $\begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix}, \begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix}$ are 2 – rowed minors of A.

Rank of a matrix

Let A be a non-zero matrix. A positive integer r is said to be the ***rank of A*** if

i) there exists a non zero r-rowed minor of A
ii) every $(r+1)$-rowed minor of A (if exists) is zero

That is, the rank of A is the order of the largest non-vanishing minor.

The rank of a zero matrix is defined to be zero.

The rank of A is denoted by $\boldsymbol{\rho(A)}$ or **rank $\boldsymbol{A}$**.

Example: Find the rank of the matrix $A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & -2 \\ 2 & 4 & -3 \end{bmatrix}$.

Solution: The matrix A is a square matrix of order 3. The only minor of A of largest order is $|A|$ of order 3. Note that

$$|A| = \begin{vmatrix} 2 & 1 & -1 \\ 0 & 3 & -2 \\ 2 & 4 & -3 \end{vmatrix} \text{ applying } \boldsymbol{R_3 \to R_3 - R_2}$$

$$= \begin{vmatrix} 2 & 1 & -1 \\ 0 & 3 & -2 \\ 0 & 3 & -2 \end{vmatrix} = 2\begin{vmatrix} 3 & -2 \\ 3 & -2 \end{vmatrix} = 0$$

Therefore, $\rho(A) \neq 3$ and so $\rho(A) < 3$.

Now we consider 2 – rowed minors of A. Note that $\begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix} = 6 \neq 0$. By definition $\rho(A) = 2$.

Example: Find the rank of the matrix $A = \begin{bmatrix} 1 & 2 & -3 & -4 \\ -2 & -4 & 6 & 8 \\ 3 & 6 & -9 & -12 \end{bmatrix}$

Solution: The given matrix is of order 3×4. Therefore it has four submatrices of order 3. They are

$$A_1 = \begin{bmatrix} 1 & 2 & -3 \\ -2 & -4 & 6 \\ 3 & 6 & -9 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 2 & -4 \\ -2 & -4 & 8 \\ 3 & 6 & -12 \end{bmatrix}, A_3 = \begin{bmatrix} 1 & -3 & -4 \\ -2 & 6 & 8 \\ 3 & -9 & -12 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} 2 & -3 & -4 \\ -4 & 6 & 8 \\ 6 & -9 & -12 \end{bmatrix}$$

Note that $|A_1| = \begin{vmatrix} 1 & 2 & -3 \\ -2 & -4 & 6 \\ 3 & 6 & -9 \end{vmatrix} = 0$ and $|A_2| = |A_3| = |A_4| = 0$.

Thus, $\rho(A) \neq 3$ and so $\rho(A) < 3$.

We now compute the 2 – rowed minors of A.

Note that a square submatrix of order 2 from A is obtained by deleting one row and 2 columns. Thus, we get ${}^3C_1 \times {}^4C_2 = 12$ submatrices of order 2. We note that each submatrix of order 2 is singular, i.e., each 2 – rowed minor of A is zero. Therefore, $\rho(A) \neq 2$. Thus $\rho(A) = 1$ (since A is not a zero matrix).

Note 1: Suppose that A is a non zero matrix of order 3. Then

i) If A is non singular then $\rho(A) = 3$
ii) If A is singular and if there is at least one of its submatrix of order 2 of A is non singular then $\rho(A) = 2$
iii) If A is singular and every submatrix of order 2 of A is singular then $\rho(A) = 1$.

Note 2: Suppose A is a matrix of order 3×4 or 4×3. The rank of A is the maximum of ranks of all submatrices of order 3 of A. (i.e., $\rho(A) \leq 3$).

Note 3: Suppose A is a matrix of order $m \times n$ or $n \times m$ then $\rho(A) \leq min(m, n)$.

Note 4: If A is a nonsingular square matrix of order n then $\rho(A) = n$

Echelon form: A matrix A is said to be in ***echelon form*** if

(i) Every row of A which has all its elements 0 occurs below every row which has a non zero element
(ii) The number of zeros before the first non-zero element in a row is less than the number of such zeros in the next row.

Example: If $A = \begin{bmatrix} 1 & 2 & -1 & -3 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ then A is in echelon form.

Note: The rank of a matrix A is equal to the number of non zero rows in echelon form of A.

Equivalence of matrices

A matrix A is said to be ***equivalent*** to another matrix B, written as $A \sim B$ if B can be obtained from A by applying finite number of elementary transformations on A.

Note: The relation $\sim$ (is equivalent to) is an equivalence relation on the set of all $m \times n$ matrices with entries from the set of real numbers.

Elementary transformations enable us to transform a given matrix into echelon form. In an echelon form, finding the highest order non singular sub matrix is easy. The following theorem enables us to find the rank of a matrix using elementary transformations.

Theorem 1: ***Elementary transformations on a matrix do not change its rank.***

Example: Find the rank of $A = \begin{bmatrix} 4 & 3 & 0 & -2 \\ 3 & 4 & -1 & -3 \\ -7 & -7 & 1 & 5 \end{bmatrix}$ using elementary transformations.

Solution:

$$A = \begin{bmatrix} 4 & 3 & 0 & -2 \\ 3 & 4 & -1 & -3 \\ -7 & -7 & 1 & 5 \end{bmatrix} \quad \text{applying } \boldsymbol{R_1 \to R_1 - R_2}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 1 \\ 3 & 4 & -1 & -3 \\ -7 & -7 & 1 & 5 \end{bmatrix} \quad \text{applying } \boldsymbol{R_2 \to R_2 - 3R_1, R_3 \to R_3 + 7R_1}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 7 & -4 & -6 \\ 0 & -14 & 8 & 12 \end{bmatrix} \quad \text{applying } \boldsymbol{R_3 \to R_3 + 2R_2}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 7 & -4 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{echelon form of } A$$

$\rho(A) =$ The number of non zero rows in the echelon form of $A = 2$.

Example: Find the rank of the matrix $A = \begin{bmatrix} 3 & -2 & 0 & -1 & -7 \\ 0 & 2 & 2 & 1 & -5 \\ 1 & -2 & -3 & -2 & 1 \\ 0 & 1 & 2 & 1 & -6 \end{bmatrix}$ using elementary transformations.

Solution:

$$A = \begin{bmatrix} 3 & -2 & 0 & -1 & -7 \\ 0 & 2 & 2 & 1 & -5 \\ 1 & -2 & -3 & -2 & 1 \\ 0 & 1 & 2 & 1 & -6 \end{bmatrix} \quad \text{applying } \boldsymbol{R_1 \leftrightarrow R_3}$$

$$\sim \begin{bmatrix} 1 & -2 & -3 & -2 & 1 \\ 0 & 2 & 2 & 1 & -5 \\ 3 & -2 & 0 & -1 & -7 \\ 0 & 1 & 2 & 1 & -6 \end{bmatrix} \quad \text{applying } \boldsymbol{R_3 \to R_3 - 3R_1}$$

$$\sim \begin{bmatrix} 1 & -2 & -3 & -2 & 1 \\ 0 & 2 & 2 & 1 & -5 \\ 0 & 4 & 9 & 5 & -10 \\ 0 & 1 & 2 & 1 & -6 \end{bmatrix} \quad \text{applying } \boldsymbol{R_2 \leftrightarrow R_4}$$

$$\sim \begin{bmatrix} 1 & -2 & -3 & -2 & 1 \\ 0 & 1 & 2 & 1 & -6 \\ 0 & 4 & 9 & 5 & -10 \\ 0 & 2 & 2 & 1 & -5 \end{bmatrix} \quad \text{applying } \boldsymbol{R_3 \to R_3 - 4R_2, R_4 \to R_4 - 2R_2}$$

$$\sim \begin{bmatrix} 1 & -2 & -3 & -2 & 1 \\ 0 & 1 & 2 & 1 & -6 \\ 0 & 0 & 1 & 1 & 14 \\ 0 & 0 & 0 & -2 & 7 \end{bmatrix} \text{ applying } \boldsymbol{R_4 \to R_4 + 2R_3, R_4 \to R_4 - 2R_2}$$

$$\sim \begin{bmatrix} 1 & -2 & -3 & -2 & 1 \\ 0 & 1 & 2 & 1 & -6 \\ 0 & 0 & 1 & 1 & 14 \\ 0 & 0 & 0 & 0 & 35 \end{bmatrix} \text{ echelon form of } A$$

$\therefore \rho(A) =$ The number of non zero rows in the echelon form of $A = 4$.

P1:

Solve the following simultaneous linear equations by using Cramer's rule.

$$\mathbf{4x + 3y = 16}$$
$$\mathbf{8x - 3y = -4}$$

Solution: The given linear equations can be written as $AX = B$, where

$A = \begin{bmatrix} 4 & 3 \\ 8 & -3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, B = \begin{bmatrix} 16 \\ -4 \end{bmatrix}$

Here $|A| = \begin{vmatrix} 4 & 3 \\ 8 & -3 \end{vmatrix} = -12 - 24 = -36 \neq 0$. Thus, A is non-singular.

Hence we can solve the given linear equations by using Cramer's rule

$$|A_1| = \begin{vmatrix} 16 & 3 \\ -4 & -3 \end{vmatrix} = -36 \quad ; \quad |A_2| = \begin{vmatrix} 4 & 16 \\ 8 & -4 \end{vmatrix} = -144$$

Hence by cramer's rule $x = \frac{|A_1|}{|A|} = \frac{-36}{-36} = 1;\ y = \frac{|A_2|}{|A|} = \frac{-144}{-36} = 4$

∴ The solution of the given system of equations is $x = 1, y = 4$.

P2:

Solve the following simultaneous linear equations by using Cramer's rule.

$$\mathbf{2x - y + 3z = 9}$$
$$\mathbf{x + y + z = 6}$$
$$\mathbf{x - y + z = 2}$$

Solution: The given linear equations can be written as $AX = B$, where

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 9 \\ 6 \\ 2 \end{bmatrix}$$

Here $|A| = \begin{vmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix}$ applying $\boldsymbol{R_2 \leftrightarrow R_1}$

$$= (-1) \begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 1 & -1 & 1 \end{vmatrix}$$ applying $\boldsymbol{R_2 \rightarrow R_2 - 2R_1, R_3 \rightarrow R_3 - R_1,}$

$$= (-1) \begin{vmatrix} 1 & 1 & 1 \\ 0 & -3 & 1 \\ 0 & -2 & 0 \end{vmatrix} = (-1)(2) = -2 \neq 0$$

Thus, A is non-singular.

Hence we can solve the given linear equations by using Cramer's rule

$$|A_1| = \begin{vmatrix} 9 & -1 & 3 \\ 6 & 1 & 1 \\ 2 & -1 & 1 \end{vmatrix} = -2 \quad ; \quad |A_2| = \begin{vmatrix} 2 & 9 & 3 \\ 1 & 6 & 1 \\ 1 & 2 & 1 \end{vmatrix} = -4$$

$$|A_3| = \begin{vmatrix} 2 & -1 & 9 \\ 1 & 1 & 6 \\ 1 & -1 & 2 \end{vmatrix} = -6$$

Hence by cramer's rule

$$x = \frac{|A_1|}{|A|} = \frac{-2}{-2} = 1 \quad ; \quad y = \frac{|A_2|}{|A|} = \frac{-4}{-2} = 2 \quad ; \quad z = \frac{|A_3|}{|A|} = \frac{-6}{-2} = 3$$

$\therefore$ The solution of the given system of equations is $x = 1, \ y = 2, \ z = 3.$

P3.

Solve the system of linear equations by matrix inversion method

$$\mathbf{2x + 5y = 11}$$

$$\mathbf{4x - 3y = 9}$$

Solution: The given system of linear equations is

$$\left.\begin{aligned} 2x + 5y &= 11 \\ 4x - 3y &= 9 \end{aligned}\right\} \quad \ldots (1)$$

(1) can be written in the matrix form as

$$AX = B \quad \ldots (2)$$

where $A = \begin{bmatrix} 2 & 5 \\ 4 & -3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, $B = \begin{bmatrix} 11 \\ 9 \end{bmatrix}$

Now, $|A| = \begin{vmatrix} 2 & 5 \\ 4 & -3 \end{vmatrix} = -6 - 20 - 26 \neq 0$

$\Longrightarrow A$ is a non-singular matrix. Therefore A is invertible and the system (2) has a unique solution $X = A^{-1}B$

We have $A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = -\frac{1}{26}\begin{bmatrix} -3 & -5 \\ -4 & 2 \end{bmatrix}$

We have $X = A^{-1}B$

i.e., $\begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{26}\begin{bmatrix} -3 & -5 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 11 \\ 9 \end{bmatrix} = -\frac{1}{26}\begin{bmatrix} -33 - 45 \\ -44 + 18 \end{bmatrix} = -\frac{1}{26}\begin{bmatrix} -78 \\ -26 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \Longrightarrow x = 3,\ y = 1$

Hence, the unique solution of the given system of linear equations is

$$x = 3\,,\ y = 1$$

P4.

Solve the system of linear equations by matrix inversion method

$$\mathbf{3x + 4y + 5z = 18}$$

$$\mathbf{2x - y + 8z = 13}$$

$$\mathbf{5x - 2y + 7z = 20}$$

Solution: The given system of linear equations is

$$\left.\begin{array}{r} 3x + 4y + 5z = 18 \\ 2x - y + 8z = 13 \\ 5x - 2y + 7z = 20 \end{array}\right\} \quad \text{... (1)}$$

(1) can be written in the matrix form as

$$AX = B \quad \text{... (2)}$$

where $A = \begin{bmatrix} 3 & 4 & 5 \\ 2 & -1 & 8 \\ 5 & -2 & 7 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 18 \\ 13 \\ 20 \end{bmatrix}$

Now, $|A| = \begin{vmatrix} 3 & 4 & 5 \\ 2 & -1 & 8 \\ 5 & -2 & 7 \end{vmatrix}$ (expanding along 1st row)

$$= 3(-7 + 16) - 4(14 - 40) + 5(-4 + 5) = 136 \neq 0$$

$\Rightarrow A$ is a non-singular matrix. Therefore A is invertible and the system (2) has a unique solution $X = A^{-1}B$

Let A_{ij} be the cofactor of a_{ij} in $= (a_{ij})_{3\times3}$, then we have

$A_{11} = (-1)^{1+1}\begin{vmatrix} -1 & 8 \\ -2 & 7 \end{vmatrix} = 9$; $A_{12} = (-1)^{1+2}\begin{vmatrix} 2 & 8 \\ 5 & 7 \end{vmatrix} = 26$

$A_{13} = (-1)^{1+3}\begin{vmatrix} 2 & -1 \\ 5 & -2 \end{vmatrix} = 1$; $A_{21} = (-1)^{2+1}\begin{vmatrix} 4 & 5 \\ -2 & 7 \end{vmatrix} = -38$

$A_{22} = (-1)^{2+2}\begin{vmatrix} 3 & 5 \\ 5 & 7 \end{vmatrix} = -4$; $A_{23} = (-1)^{2+3}\begin{vmatrix} 3 & 4 \\ 5 & -2 \end{vmatrix} = 26$

$A_{31} = (-1)^{3+1}\begin{vmatrix} 4 & 5 \\ -1 & 8 \end{vmatrix} = 37$; $A_{32} = (-1)^{3+2}\begin{vmatrix} 3 & 5 \\ 2 & 8 \end{vmatrix} = -14$

$A_{33} = (-1)^{3+3}\begin{vmatrix} 3 & 4 \\ 2 & -1 \end{vmatrix} = -1$

The matrix of cofactors of A is

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} -9 & 8 & -5 \\ -8 & 7 & -4 \\ -2 & 2 & -1 \end{bmatrix}$$

The adjoint of A is the transpose of the matrix of cofactors of A.

$$adjA = \begin{bmatrix} 9 & 26 & 1 \\ -38 & -4 & 26 \\ 37 & -14 & -11 \end{bmatrix}^T = \begin{bmatrix} 9 & -38 & 37 \\ 26 & -4 & -14 \\ 1 & 26 & -11 \end{bmatrix}$$

We have $A^{-1} = \frac{1}{|A|}adjA = \frac{1}{136}\begin{bmatrix} 9 & -38 & 37 \\ 26 & -4 & -14 \\ 1 & 26 & -11 \end{bmatrix}$

We have, $X = A^{-1}B$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{136}\begin{bmatrix} 9 & -38 & 37 \\ 26 & -4 & -14 \\ 1 & 26 & -11 \end{bmatrix}\begin{bmatrix} 18 \\ 13 \\ 20 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

$\Rightarrow x = 3,\ y = 1,\ z = 1$

Hence, the unique solution of the given system of linear equations is

$$x = 3,\ y = 1,\ z = 1$$

IP1:

Solve the following simultaneous linear equations by using Cramer's rule.

$$\mathbf{3x - 2y = -4}$$
$$\mathbf{x + y = 12}$$

Solution: The given linear equations can be written as $AX = B$, where

$A = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, B = \begin{bmatrix} -4 \\ 12 \end{bmatrix}$

Here $|A| = \begin{vmatrix} 3 & -2 \\ 1 & 1 \end{vmatrix} = 3 + 2 = 5 \neq 0$. Thus, A is non-singular.

Hence we can solve the given linear equations by using Cramer's rule

$$|A_1| = \begin{vmatrix} -4 & -2 \\ 12 & 1 \end{vmatrix} = 20 \quad ; \quad |A_2| = \begin{vmatrix} 3 & -4 \\ 1 & 12 \end{vmatrix} = 40$$

Hence by cramer's rule $x = \frac{|A_1|}{|A|} = \frac{20}{5} = 4$; $y = \frac{|A_2|}{|A|} = \frac{40}{5} = 8$

∴ The solution of the given system of equations is $x = 4, y = 8$.

IP2:

Solve the following simultaneous linear equations by using Cramer's rule.

$$\mathbf{3x + 4y + 5z = 18}$$
$$\mathbf{2x - y + 8z = 13}$$
$$\mathbf{5x - 2y + 7z = 20}$$

Solution: The given linear equations can be written as $AX = B$, where

$$A = \begin{bmatrix} 3 & 4 & 5 \\ 2 & -1 & 8 \\ 5 & -2 & 7 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 18 \\ 13 \\ 20 \end{bmatrix}$$

Here $|A| = \begin{vmatrix} 3 & 4 & 5 \\ 2 & -1 & 8 \\ 5 & -2 & 7 \end{vmatrix}$ (expanding along first row)

$$= 3(-7 + 16) - 4(14 - 40 + 5(-4 + 5) = 136 \neq 0$$

Thus, A is non-singular.

Hence we can solve the given linear equations by using Cramer's rule

$$|A_1| = \begin{vmatrix} 18 & 4 & 5 \\ 13 & -1 & 8 \\ 20 & -2 & 7 \end{vmatrix} = 408 \quad ; \quad |A_2| = \begin{vmatrix} 3 & 18 & 5 \\ 2 & 13 & 8 \\ 5 & 20 & 7 \end{vmatrix} = 136$$

$$|A_3| = \begin{vmatrix} 3 & 4 & 18 \\ 2 & -1 & 13 \\ 5 & -2 & 20 \end{vmatrix} = 136$$

Hence by cramer's rule

$$x = \frac{|A_1|}{|A|} = \frac{408}{136} = 3 \quad ; \quad y = \frac{|A_2|}{|A|} = \frac{136}{136} = 1 \quad ; \quad z = \frac{|A_3|}{|A|} = \frac{136}{136} = 1$$

$\therefore$ The solution of the given system of equations is $x = 3,\ y = 1,\ z = 1.$

IP3.

Solve the system of linear equations by matrix inversion method

$$\boldsymbol{x - y + z = 2}$$

$$\boldsymbol{2x - y = 0}$$

$$\boldsymbol{2y - z = 1}$$

Solution: The given system of linear equations is

$$\left.\begin{array}{r} x - y + z = 2 \\ 2x - y = 0 \\ 2y - z = 1 \end{array}\right\} \quad \ldots (1)$$

(1) can be written in the matrix form as

$$AX = B \quad \ldots (2)$$

where $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 0 & 2 & -1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ and

Now, $|A| = \begin{vmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 0 & 2 & -1 \end{vmatrix}$ (expanding along 1^{st} column)

$$= 1(1-0) - 2(1-2) = 1 + 2 = 3 \neq 0$$

$\Rightarrow A$ is a non-singular matrix. Therefore A is invertible and the system (2) has a unique solution $X = A^{-1}B$

Let A_{ij} be the cofactor of a_{ij} in $= (a_{ij})_{3\times3}$, then we have

$A_{11}(-1)^{1+1}\begin{vmatrix} -1 & 0 \\ 2 & -1 \end{vmatrix} = 1$; $A_{12} = (-1)^{1+2}\begin{vmatrix} 2 & 0 \\ 0 & -1 \end{vmatrix} = 2$

$A_{13} = (-1)^{1+3}\begin{vmatrix} 2 & -1 \\ 0 & 2 \end{vmatrix} = 4$; $A_{21} = (-1)^{2+1}\begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix} = 1$

$A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} = -1$; $A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix} = -2$

$A_{31} = (-1)^{3+1}\begin{vmatrix} -1 & 1 \\ -1 & 0 \end{vmatrix} = 1$; $A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} = 2$

$A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} = 1$

The matrix of cofactors of A is

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 1 & -1 & -2 \\ 1 & 2 & 1 \end{bmatrix}$$

The adjoint of A is the transpose of the matrix of cofactors of A.

$$adjA = \begin{bmatrix} 1 & 2 & 4 \\ 1 & -1 & -2 \\ 1 & 2 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 2 \\ 4 & -2 & 1 \end{bmatrix}$$

We have $A^{-1} = \frac{1}{|A|} adjA = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 2 \\ 4 & -2 & 1 \end{bmatrix}$

We have, $X = A^{-1}B$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 2 \\ 4 & -2 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$\Rightarrow x = 1,\ y = 2,\ z = 3$

Hence, the unique solution of the given system of linear equations is

$$x = 1,\ y = 2,\ z = 3$$

IP4.

Solve the system of linear equations

$$\mathbf{5x - 6y + 4z = 15}$$

$$\mathbf{7x + 4y - 3z = 19}$$

$$\mathbf{2x + y + 6z = 46}$$

Solution: The given system of linear equations is

$$\left.\begin{array}{r} 5x - 6y + 4z = 15 \\ 7x + 4y - 3z = 19 \\ 2x + y + 6z = 46 \end{array}\right\} \qquad \text{... (1)}$$

(1) can be written in the matrix form as

$$AX = B \qquad \text{... (2)}$$

where $A = \begin{bmatrix} 5 & -6 & 4 \\ 7 & 4 & -3 \\ 2 & 1 & 6 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 15 \\ 19 \\ 46 \end{bmatrix}$

Now, $|A| = \begin{vmatrix} 5 & -6 & 4 \\ 7 & 4 & -3 \\ 2 & 1 & 6 \end{vmatrix}$ (expanding along 1st row)

$$= 5(24 + 3) + 6(42 + 6) + 4(7 - 8) = 419 \neq 0$$

$\Rightarrow A$ is a non-singular matrix. Therefore A is invertible and the system (2) has a unique solution $X = A^{-1}B$

Let A_{ij} be the cofactor of a_{ij} in $= (a_{ij})_{3\times3}$, then we have

$A_{11} = (-1)^{1+1}\begin{vmatrix} 4 & -3 \\ 1 & 6 \end{vmatrix} = 27$; $A_{12} = (-1)^{1+2}\begin{vmatrix} 7 & -3 \\ 2 & 6 \end{vmatrix} = -48$

$A_{13} = (-1)^{1+3}\begin{vmatrix} 7 & 4 \\ 2 & 1 \end{vmatrix} = -1$; $A_{21} = (-1)^{2+1}\begin{vmatrix} -6 & 4 \\ 1 & 6 \end{vmatrix} = 40$

$A_{22} = (-1)^{2+2}\begin{vmatrix} 5 & 4 \\ 2 & 6 \end{vmatrix} = 22$; $A_{23} = (-1)^{2+3}\begin{vmatrix} 3 & 4 \\ 5 & -2 \end{vmatrix} = 26$

$A_{31} = (-1)^{3+1}\begin{vmatrix} -6 & 4 \\ 4 & -3 \end{vmatrix} = 2$; $A_{32} = (-1)^{3+2}\begin{vmatrix} 5 & 4 \\ 7 & -3 \end{vmatrix} = 43$

$A_{33} = (-1)^{3+3}\begin{vmatrix} 5 & -6 \\ 7 & 4 \end{vmatrix} = 62$

$$C = \begin{bmatrix} 27 & -48 & -1 \\ 40 & 22 & -17 \\ 2 & 43 & 62 \end{bmatrix}$$

The matrix of cofactors of A is

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} 27 & -48 & -1 \\ 40 & 22 & -17 \\ 2 & 43 & 62 \end{bmatrix}$$

The adjoint of A is the transpose of the matrix of cofactors of A.

$$adjA = \begin{bmatrix} 27 & -48 & -1 \\ 40 & 22 & -17 \\ 2 & 43 & 62 \end{bmatrix}^T = \begin{bmatrix} 27 & 40 & 2 \\ -48 & 22 & 43 \\ -1 & -17 & 62 \end{bmatrix}$$

We have $A^{-1} = \frac{1}{|A|} adjA = \frac{1}{419}\begin{bmatrix} 27 & 40 & 2 \\ -48 & 22 & 43 \\ -1 & -17 & 62 \end{bmatrix}$

We have, $X = A^{-1}B$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{419}\begin{bmatrix} 27 & 40 & 2 \\ -48 & 22 & 43 \\ -1 & -17 & 62 \end{bmatrix}\begin{bmatrix} 15 \\ 19 \\ 46 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$$

$\Rightarrow x = 3,\ y = 4,\ z = 6$

Hence, the unique solution of the given system of linear equations is

$$x = 3,\ y = 4,\ z = 6$$

10.8. Linear Equations

EXERCISES

1. **Solve the following linear equations by Cramer's rule and Matrix Inversion method**

   a. $5x + 7y + 2 = 0$
   $4x + 6y + 3 = 0$

   b. $5x + 2y = 3$
   $3x + 2y = 5$

   c. $3x + 4y - 5 = 0$
   $x - y + 3 = 0$

   d. $3x + y = 19$
   $2x - 3y = 24$

2. **Solve the following linear equations by Cramer's rule and Matrix Inversion method**

   a. $x + y - z = 3$
   $2x + 3y + z = 10$
   $3x - y - 7z = 1$

   b. $x + y + z = 3$
   $2x - y + z = -1$
   $2x + y - 3z = -9$

   c. $6x - 12y + 25z = 4$
   $4x + 15y - 20z = 3$
   $2x + 18y + 15z = 10$

   d. $5x + 3y + z = 16$
   $2x + y + 3z = 19$
   $x + 2y + 4z = 25$

   e. $3x + 4y + 2z = 8$
   $2y - 3z = 3$
   $x - 2y + 6z = -2$

   f. $2x + y + z = 2$
   $x + 3y - z = 5$
   $3x + y - 2z = 6$

   g. $2x + 6y = 2$
   $3x - z = -8$
   $2x - y + z = -3$

   h. $3x + 4y + 7z = 14$
   $2x - y + 3z = 4$
   $x + 2y - 3z = 0$

10.10. System of Linear Equations and consistency

Consider a system of m linear equations in n unknowns $x_1, x_2, \ldots, x_n$ (where $m = n$, $m < n$, $m > n$). Such a system can be written as

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \quad \ldots(1)$$

where a_{ij} and b_i are real numbers (for $1 \le i \le m$ and $1 \le j \le n$).

If each $b_i = 0$, $1 \le i \le m$, then the system (1) is said to be a **homogeneous system**. If at least one $b_i \ne 0$, then the system (1) is said to be a **non-homogeneous system**.

System (1) can be represented by the matrix equation

$$AX = B \quad \ldots(2)$$

where $A = [a_{ij}]_{m\times n}$, $X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and $B = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$

The matrix A is called the **coefficient matrix** and the matrix $[A\ B]$ (the matrix obtained by placing the column matrix B to the right of the matrix A) is called the **augmented matrix**.

$$[A\ B] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

A set of values $x_1, x_2, \ldots, x_n$ satisfying the equations in the system (1) is called a **solution** of the system (1).

The system (1) is said to be **consistent** if the system has a solution. If the system has no solutions then it is said to be **inconsistent**.

The solution of the system of linear equations (2) depends on the ranks of the coefficient matrix A and the augmented matrix $[A\ B]$. The following theorem gives the nature of solutions of the system.

Theorem 1: The system of m linear equations in n unknowns

$$AX = B \quad \ldots(2)$$

is consistent if and only if $\rho(A) = \rho[A\ B]$.

Further, the system (2) has

(i) unique solution if $\rho(A) = \rho[A\ B] = n$ (the number of unknowns)
(ii) infinitely many solutions if $\rho(A) = \rho[A\ B] < n$

Note:

1. The system (2) is inconsistent if $\rho(A) \ne \rho[A\ B]$.
2. Suppose $m = n$, i.e., the number of equations is equal to the number of unknowns. Then in $AX = B$, A is a square matrix of order n. If A is singular then we cannot apply Cramer's rule and matrix inversion method for the solution of the system. But we can apply this method. Further, we can apply this method even if A is non-singular.
3. If $m < n$ and the system is consistent then it has infinitely many solutions.
4. We use only elementary row transformation to reduce $[A\ B]$ to an echelon form.

Example: Examine whether the following system of equations is consistent or inconsistent:

[illegible]

Example: Examine whether the following system of equations is consistent or inconsistent:

[illegible]

Example: Examine whether the following system of equations is consistent or inconsistent:

[illegible]

$$[A\ B] = \begin{bmatrix} \ldots \\ 3 & 0 & -2 & 3 & -4 \\ 5 & -4 & 0 & 1 & -3 \end{bmatrix} \text{ applying } R_1 \to R_1 - R_2$$

$$\sim \begin{bmatrix} 1 & -2 & 1 & -1 & -1 \\ 3 & 0 & -2 & 3 & -4 \\ 5 & -4 & 0 & 1 & -3 \end{bmatrix} \text{ applying } R_2 \to R_2 - 3R_1, R_3 \to R_3 - 5R_1$$

$$\sim \begin{bmatrix} 1 & -2 & 1 & -1 & -1 \\ 0 & 6 & -5 & 6 & -1 \\ 0 & 6 & -5 & 6 & 2 \end{bmatrix} \text{ applying } R_3 \to R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & -2 & 1 & -1 & -1 \\ 0 & 6 & -5 & 6 & -1 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix} \text{ echelon form of } [A\ B] \quad \ldots(3)$$

Now, $\rho[A\ B]$ = The number of non zero rows in the echelon form of $[A\ B]$

$= 3$

And $\rho(A) = \text{rank of} \begin{bmatrix} 1 & -2 & 1 & -1 \\ 0 & 6 & -5 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} = 2$

Thus $\rho(A) \ne \rho[A\ B]$. Therefore, by theorem 1 the given system is inconsistent.

Example: Investigate for what values of λ and μ the simultaneous equations

$$x + y + z = 6$$
$$x + 2y + 3z = 10$$
$$x + 2y + \lambda z = \mu$$

have i) no solution ii) unique solution iii) infinitely many solutions

Solution: The given system is a non-homogeneous system of three equations in three unknowns.

The matrix equation of the given system of equations is $AX = B$, where

$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$

The augmented matrix is $[A\ B]$

$$[A\ B] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{bmatrix} \text{ applying } R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{bmatrix} \text{ applying } R_3 \to R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{bmatrix}$$

i. If $\lambda - 3 \ne 0$, then $\rho(A) = \rho([A\ B]) = 3$ = number of unknowns. Therefore the system has a unique solution.

ii. If $\lambda - 3 = 0$ and $\mu - 10 = 0$ i.e., $\lambda = 3$ and $\mu = 10$, then

$\rho(A) = \rho([A\ B]) = 2$, which is less than the number of unknowns.

Therefore, the system has infinitely many solutions.

iii. If $\lambda - 3 = 0$ and $\mu - 10 \ne 0$ i.e., $\lambda = 3$ and $\mu \ne 10$, then

$\rho(A) = 2$, $\rho([A\ B]) = 3$, i.e., $\rho(A) \ne \rho([A\ B])$. Therefore the system is inconsistent.

Homogeneous system of linear equations.

Consider the following system of m homogeneous linear equations in n unknowns

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned} \quad \ldots(1)$$

The matrix equation of the above system is

$$AX = O \quad \ldots(2)$$

where $A = [a_{ij}]_{m\times n}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $O = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{m\times 1}$

Note that the coefficient matrix A and the augmented matrix $[A\ O]$ have the same rank (because they differ by a column of zeros). Therefore, a **homogeneous system of linear equations is always consistent.** We see that

$$x_1 = x_2 = x_3 = \cdots = x_n = 0$$

is always a solution. This solution is called the **trivial solution**. A solution which is not trivial solution is called a nontrivial solution. We are interested in finding a nontrivial solutions of a homogeneous system of linear equations.

The following theorem gives the nature of solutions of the system (2)

Theorem 2:

The system of m homogeneous linear equations in n unknowns $AX = O$ has

a. the trivial solution only if $\rho(A) = n$ (the number of unknowns)
b. an infinite number of solutions if $\rho(A) < n$

Note: If $m < n$ (the number of equations is less than the number of unknowns) then a system of m homogeneous linear equations in n unknowns always has an infinite number of nontrivial solutions

Example: Find all the nontrivial solutions, if any, for the following linear system of equations:

$$x_1 - x_2 + x_3 = 0$$
$$x_1 + 2x_2 + x_3 = 0$$
$$2x_1 + x_2 + 3x_3 = 0$$

Solution: The given system is a homogeneous system of three linear equations in three unknowns. The matrix equation of the given system of equations is $AX = O$,

where $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\text{Now, } A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} \text{ applying } R_2 \to R_2 - R_1, R_3 \to R_3 - 2R_1$$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & 0 \\ 0 & 3 & 1 \end{bmatrix} \text{ applying } R_3 \to R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ echelon form of } A$$

and $\rho(A)$ = the number of non zero rows of A

$= 3$ = the number of unknowns

Therefore, by theorem 2, the trivial solution $x_1 = x_2 = x_3 = 0$ is the only solution of the given system.

Example:

Find all nontrivial solutions, if any, for the following system of equations:

$$2x_1 - 4x_2 - 5x_3 = 0$$
$$3x_1 - 6x_2 + x_3 = 0$$
$$5x_1 - 10x_2 - 3x_3 = 0$$

Solution: The given system is a homogeneous system of four linear equations in three unknowns. The matrix equation of the given system of equations is $AX = O$,

where $A = \begin{bmatrix} 2 & -4 & -5 \\ 3 & -6 & 1 \\ 5 & -10 & -3 \\ 7 & -14 & -16 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, O = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$$\text{Now, } A = \begin{bmatrix} 2 & -4 & -5 \\ 3 & -6 & 1 \\ 5 & -10 & -3 \\ 7 & -14 & -16 \end{bmatrix} \text{ applying } R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 3 & -6 & 1 \\ 2 & -4 & -5 \\ 5 & -10 & -3 \\ 7 & -14 & -16 \end{bmatrix} \text{ applying } R_1 \to R_1 - R_2$$

$$\sim \begin{bmatrix} 1 & -2 & 6 \\ 2 & -4 & -5 \\ 5 & -10 & -3 \\ 7 & -14 & -16 \end{bmatrix} \text{ applying } R_2 \to R_2 - 2R_1, R_3 \to R_3 - 5R_1, R_4 \to R_4 - 7R_1$$

$$\sim \begin{bmatrix} 1 & -2 & 6 \\ 0 & 0 & -17 \\ 0 & 0 & -33 \\ 0 & 0 & -58 \end{bmatrix} \text{ applying } R_2 \to \left(-\tfrac{1}{17}\right)R_2, R_3 \to \left(-\tfrac{1}{33}\right)R_3, R_4 \to \left(-\tfrac{1}{58}\right)R_4$$

$$\sim \begin{bmatrix} 1 & -2 & 6 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \text{ applying } R_3 \to R_3 - R_2, R_4 \to R_4 - R_2$$

$$\sim \begin{bmatrix} 1 & -2 & 6 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ echelon form of } A$$

and $\rho(A) = 2 < 3$ (the number of unknowns). Therefore, by theorem 2, the system has an infinite number of nontrivial solutions. We have the following equivalent system of equations

$$x_1 - 2x_2 + 6x_3 = 0$$
$$x_3 = 0$$

i.e., $x_3 = 0$ and $x_1 = 2x_2$

If $x_2 = k \in \mathbb{R}$ then $x_1 = 2k$, $x_3 = 0$

Thus $x_1 = 2k, x_2 = k, x_3 = 0$, where $k \in \mathbb{R}$ is arbitrary. If $k \ne 0$, then we get an infinite number of nontrivial solutions.

Example:

Find all nontrivial solutions, if any, for the following system of equations:

$$2x_1 + 3x_2 - x_3 + x_4 = 0$$
$$3x_1 + 2x_2 - 2x_3 + 2x_4 = 0$$
$$5x_1 - 4x_3 + 4x_4 = 0$$

Solution: The given system is a homogeneous system of three linear equations in four unknowns. Since the number of equations is less than the number of unknowns, the system has an infinite number of nontrivial solutions.

The matrix equation of the given system of equations is $AX = O$,

Where $A = \begin{bmatrix} 2 & 3 & -1 & 1 \\ 3 & 2 & -2 & 2 \\ 5 & 0 & -4 & 4 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\text{Now, } A = \begin{bmatrix} 2 & 3 & -1 & 1 \\ 3 & 2 & -2 & 2 \\ 5 & 0 & -4 & 4 \end{bmatrix} \text{ applying } R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 3 & 2 & -2 & 2 \\ 2 & 3 & -1 & 1 \\ 5 & 0 & -4 & 4 \end{bmatrix} \text{ applying } R_1 \to R_1 - R_2$$

$$\sim \begin{bmatrix} 1 & -1 & -1 & 1 \\ 2 & 3 & -1 & 1 \\ 5 & 0 & -4 & 4 \end{bmatrix} \text{ applying } R_2 \to R_2 - 2R_1, R_3 \to R_3 - 5R_1$$

$$\sim \begin{bmatrix} 1 & -1 & -1 & 1 \\ 0 & 5 & 1 & -1 \\ 0 & 5 & 1 & -1 \end{bmatrix} \text{ applying } R_3 \to R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & -1 & -1 & 1 \\ 0 & 5 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ echelon form of } A$$

and $\rho(A) = 2 < 4$ (the number of unknowns). Therefore, by theorem 2, the system has an infinite number of nontrivial solutions. We have the following equivalent system of equations.

$$x_1 - x_2 - x_3 + x_4 = 0$$
$$5x_2 + x_3 - x_4 = 0$$

i.e., $5x_2 = -x_3 + x_4$ and $x_1 - x_2 + 5x_2 = 0 \Rightarrow x_1 = -4x_2$

If $x_3 = \lambda \in \mathbb{R}$ and $x_4 = \mu \in \mathbb{R}$, then $x_2 = -\frac{1}{5}\lambda + \frac{1}{5}\mu$ and $x_1 = -4\left(-\frac{1}{5}\lambda + \frac{1}{5}\mu\right)$

If $(\lambda, \mu) \ne (0, 0)$, then we get the following infinite number of solutions:

$x_1 = \frac{4}{5}\lambda - \frac{4}{5}\mu, -\frac{1}{5}\lambda + \frac{1}{5}\mu, x_3 = \lambda, x_4 = \mu$, where λ and μ are arbitrary constants.

10.10. System of Linear Equations and consistency

Exercise:

I. **Examine whether the following system of equations is consistent or inconsistent .If it is consistent, find the complete solution.**

1. $x_1 + 2x_2 = 1, -3x_1 + 2x_2 = -2, -x_1 + 6x_2 = 0$
2. $2x_1 - x_2 + x_3 = 4, 3x_1 - x_2 + x_3 = 6, 4x_1 - x_2 + 2x_3 = 7, -x_1 + x_2 - x_3 = 9$
3. $x_1 - x_2 + x_3 = 2, 3x_1 - x_2 + 2x_3 = -6, 3x_1 + x_2 + x_3 = -18$
4. $x_1 + 2x_2 + x_3 = 2, 2x_1 + 4x_2 + 3x_3 = 3, 3x_1 + 6x_2 + 5x_3 = 4$
5. $2x + 6y = -11, 6x + 20y - 6z = -3, 6y - 18z = -1$
6. $x + y + z = 9, 2x + 5y + 7z = 52, 2x + y - z = 0$

II. **Find all the non trivial solutions, if any , for the following system of equations**

7. $3x + y - 4z = 0, 2x + 5y + 6z = 0, x - 3y - 8z = 0$
8. $2x + y + 3z = 0, x + 2y - z = 0, x - y + z = 0$
9. $x + y - z + t = 0, x - y + 2z - t = 0, 3x + y + t = 0$
10. $2w + 3x - y - z = 0, 4w - 6x - 2y + 2z = 0, -6w + 12x - 3y - 4z = 0$
11. $2x + 3y - z = 0,\ x - y - 2z = 0, 3x + y + 3z = 0$

Answers:

I.

1. Consistent , unique solution, $x_1 = \frac{3}{4}, x_2 = \frac{1}{8}$
2. Inconsistent
3. Consistent, Infinitely many solutions, $x_1 = -\frac{1}{2}k - 4, x_2 = \frac{1}{2}k - 6, x_3 = k \in \boldsymbol{R}$
4. Consistent , infinitely many solutions, $x_1 = 3 - 2k, x_2 = k, x_3 = -1, k \in \boldsymbol{R}$
5. Inconsistent
6. Consistent , unique solution, $x = 1, y = 3, z = 5$

II.

7. $x = 2k, y = -2k, z = k, k \in \boldsymbol{R}$, infinitely many nontrivial solutions
8. $x = y = z = 0$
9. $x = -\frac{\lambda}{2}, y = \frac{3}{2}\lambda - \mu, z = \lambda, t = \mu,\ \lambda, \mu \in \boldsymbol{R}$, infinitely many nontrivial solutions
10. $x = \frac{\lambda}{3}, y = \mu, z = \lambda, w = \frac{\mu}{2}, \lambda, \mu \in \boldsymbol{R}$, infinitely many nontrivial solutions
11. Trivial solution only